

ON TWO DIMENSIONAL WEIGHT TWO ODD REPRESENTATIONS OF TOTALLY REAL FIELDS

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ABSTRACT. We say that a two dimensional p -adic Galois representation $G_F \rightarrow \mathrm{GL}_2(\overline{\mathbb{Q}}_p)$ of a number field F is *weight two* if it is de Rham with Hodge-Tate weights 0 and -1 equally distributed at each place above p ; for example, the Tate module of an elliptic curve has this property. The purpose of this paper is to establish a variety of results concerning odd weight two representations of totally real fields in as great a generality as we are able. Most of these results are improvements upon existing results. Three of our main results are as follows. (1) We prove a modularity lifting theorem for odd weight two representations, extending a theorem of Kisin to include representations which are not potentially crystalline. (2) We show that essentially any odd weight two representation is potentially modular, following the ideas of Taylor. (3) We show that one can lift essentially any odd residual representation to a minimally ramified weight two p -adic representation, using some ideas of Khare-Wintenberger. As an application of these results we show that if ρ is a sufficiently irreducible odd weight two p -adic representation of a totally real field F and either F has odd degree or ρ is indecomposable at some finite place $v \nmid p$ then ρ occurs as the Tate module of a GL_2 -type abelian variety. This establishes some new cases of the Fontaine-Mazur conjecture.

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1. INTRODUCTION

(1.1) Fix a totally real field F and a prime p . Denote by G_F the absolute Galois group of F . We say that a representation $G_F \rightarrow \mathrm{GL}_2(\overline{\mathbb{Q}}_p)$ is *weight two* if it is de Rham with non-positive Hodge-Tate weights at each place above p and has determinant equal to a finite order character times the cyclotomic character; see §1.2 for more on this definition. The purpose of this paper is to establish a variety of results concerning odd weight two representations of totally real fields in as great a generality as we are able. These results are, for the most part, extensions of the work of Khare, Kisin, Taylor and Wintenberger.

For the purposes of the introduction, assume $p \neq 2, 5$. We prove many of the results outlined below when $p = 5$ with an additional hypothesis; we do not consider $p = 2$ in this paper. One of our main results is the following theorem:

Theorem 1.1.1. *Let $\rho : G_F \rightarrow \mathrm{GL}_2(\overline{\mathbb{Q}}_p)$ be a finitely ramified, odd, weight two representation such that $\bar{\rho}|_{G_F(\zeta_p)}$ is irreducible. Then ρ is potentially modular.*

Here “finitely ramified” means ρ ramifies at only finitely many places; “odd” means $\det \rho(c) = -1$ for any complex conjugation c ; $\bar{\rho}$ denotes the mod p reduction of ρ ; and “potentially modular” means that there

is a finite, totally real extension F'/F such that $\rho|_{G_{F'}}$ is associated to a *cuspidal* Hilbert eigenform. (All modular forms we use will be cuspidal.) We actually prove a more precise version of Theorem 1.1.1 — see Theorem 5.1.2. Using known consequences of potential modularity we obtain the following:

Corollary 1.1.2. *Let ρ be as in the theorem.*

- (1) *If ρ is unramified at v then the eigenvalues of $\rho(\text{Frob}_v)$ belong to $\overline{\mathbb{Q}} \subset \overline{\mathbb{Q}}_p$ and under any embedding into \mathbb{C} have modulus $(Nv)^{1/2}$.*
- (2) *The representation ρ fits into a strongly compatible system.*
- (3) *For any isomorphism $i : \overline{\mathbb{Q}}_p \rightarrow \mathbb{C}$ the L -function $L(s, \rho, i)$ converges to a non-zero holomorphic function for $\text{Re } s > \frac{3}{2}$, has a meromorphic continuation to the entire complex plane and satisfies the expected functional equation.*
- (4) *Assume that either F has odd degree or for some place $v \nmid p$ the representation $\rho|_{G_{F_v}}$ is indecomposable. Then ρ occurs as the Tate module of a GL_2 -type abelian variety. In particular, the Fontaine-Mazur conjecture holds for ρ .*

This corollary is proved in §9. The additional hypothesis in the final statement stems from the fact that the Galois representation attached to a parallel weight two Hilbert eigenform over a totally real field is known to come from an abelian variety only if the field has odd degree or the form is square integrable at some finite place.

Another of our main results is the following:

Theorem 1.1.3. *Let $\overline{\rho} : G_F \rightarrow \text{GL}_2(\overline{\mathbb{F}}_p)$ be an irreducible odd representation.*

- (1) *Assume $\overline{\rho}|_{G_{F(\zeta_p)}}$ is irreducible. Then there exists a minimally ramified weight two lift of $\overline{\rho}$ to $\overline{\mathbb{Q}}_p$.*
- (2) *The representation $\overline{\rho}$ occurs as the p -torsion of a GL_2 -type abelian variety.*

The first statement is restated and proved as Proposition 7.10.1; the second is restated and proved as Proposition 9.4.2. In §7 we will prove a result considerably stronger than the first statement, showing that one has great control over the local behavior of lifts: one can construct lifts with a given action of inertia at finitely many places, so long as there are no obvious obstructions. In Remark 9.4.5 we explain how one can often remove the additional hypothesis in the first statement.

We mention one more of our results here, a modularity lifting theorem (restated and proved as Theorem 4.4.2):

Theorem 1.1.4. *Let $\rho : G_F \rightarrow \text{GL}_2(\overline{\mathbb{Q}}_p)$ be a finitely ramified, odd, weight two representation such that $\overline{\rho}|_{G_{F(\zeta_p)}}$ is irreducible. Assume that there exists a parallel weight two Hilbert eigenform f such that $\overline{\rho} = \overline{\rho}_f$ and that ρ and ρ_f have the same “type” at each place above p , where “type” is one of: potentially crystalline ordinary, potentially crystalline non-ordinary or not potentially crystalline. Then there is a Hilbert eigenform g such that $\rho = \rho_g$.*

(1.2) We now give a precise definition of the term “weight two.” Let F_v/\mathbb{Q}_p be a finite extension and let E/\mathbb{Q}_p be any extension. We say that a representation $\rho_v : G_{F_v} \rightarrow \text{GL}_2(E)$ is *weight two* if the $(E \otimes_{\mathbb{Q}_p} F_v)$ -module $(\rho \otimes_{\mathbb{Q}_p} B_{\text{dR}})^{G_{F_v}}$ is free of rank two and its associated graded is free of rank one in degrees 0 and -1 . Equivalently, ρ_v is weight two if it is de Rham with non-positive Hodge-Tate weights and $\det \rho_v$ restricted to inertia is a finite order character times the cyclotomic character. We use the convention that the cyclotomic character χ_p has Hodge-Tate weight -1 .

For a global field F/\mathbb{Q} , we say that a representation $\rho : G_F \rightarrow \text{GL}_2(E)$ is *weight two* if it is at every place above p . Equivalently, ρ is weight two if $\rho|_{G_{F_v}}$ is de Rham with non-positive Hodge-Tate weights for each $v \mid p$ and $\det \rho$ is equal to a finite order character times the cyclotomic character. We use this terminology because the Galois representations coming from parallel weight two Hilbert eigenforms have this property.

(1.3) We now give an overview of the structure of the paper.

- In §2 we give some results on deformation rings of representations of local Galois groups. These results are mostly due to Kisin [Ki], though we will also need some results of Gee [Ge] and some results that do not seem to be covered by either author (for which we provide proofs).
- In §3 we prove an $R = \mathbb{T}$ theorem using the method of Taylor-Wiles, as improved by several authors. We follow [Ki] very closely in our treatment. This theorem relies on the local results of §2.

- In §4 we establish Theorem 1.1.4. We deduce this from the $R = \mathbb{T}$ theorem of §3 using the base change tricks of [Ki]. At the end of §4 we appeal to the results of §7 and the modularity lifting theorem of Skinner-Wiles to produce some mod p congruences between Hilbert modular forms which are special at p and those which are not.
- In §5 we prove our potential modularity results (these are somewhat stronger than the statement in Theorem 1.1.1). Such results were first established by Taylor [Tay3], and our proofs do not differ much from his. In fact, our proofs are less difficult because we have stronger modularity lifting theorems available. The main tools used in the proofs are Theorem 1.1.4 and a theorem of Moret-Bailly.
- In §6 we prove that certain global deformation rings are finite over \mathbb{Z}_p of non-zero rank, and therefore have characteristic zero points. Such results were first established by Khare and Wintenberger [KW], and our proof does not differ much from theirs. The proof has two parts. First one uses purely Galois theoretic results to show that the deformation rings have non-zero Krull dimension. Then one uses the potential modularity results of §5 together with the finiteness of deformation rings of modular representations (proved in §3) to conclude that the deformation rings are finite.
- In §7 we use the results of §6 to produce lifts of residual representations with prescribed behavior on inertia at finitely many places. We learned the method of proof from a paper of Gee [Ge2]. Our results improve on the ones there in two ways: first, we make no modularity hypothesis; and second, we can control the monodromy operator of lifts. To apply the results of §6 for these purposes, we need some more results on local deformation rings. These are due to Kisin [Ki4] and Gee [Ge2], though as before we need to establish some new results in certain easy cases. We also prove some local lifting results which do not appear to be in the literature.
- In §8 we prove two miscellaneous results which rely on all the preceding work. The first is a descent result for mod p Hilbert eigenforms. A similar result, though less general, was obtained previously by Khare [Kh]. The second result is an improvement on our potential modularity result. It states that one can achieve modularity by passing to an extension which splits at a finite prescribed set of primes. We establish this by using a simple modification of the theorem of Moret-Bailly and our descent theorem for mod p eigenforms.
- In §9 we prove Corollary 1.1.2. The proofs of the statements in this corollary are, for the most part, well-known or appear already in the literature. It seems worthwhile, though, to have them consolidated in one place under one notational scheme. Statement (1) of the corollary follows easily from the corresponding result for modular representations, which is due to Blasius [Bl]. The proof of statement (2) is due to Dieulefait [Di]. The proofs of statements (3) and (4) are due to Taylor [Tay], [Tay3]. We should point out, though, that to prove (4) in the stated generality requires some new arguments, in particular the strengthened form of potential modularity given in §8.

(1.4) We now list some notation that we will use throughout the paper.

- The letter p always denotes an odd prime. We deal with p -adic or mod p representations.
- The symbol F denotes the field over which we work. It is almost always totally real and there is usually a representation of its absolute Galois group G_F that we are studying.
- Finite extensions of \mathbb{Q}_p or \mathbb{Q}_ℓ whose Galois group we study will be denoted by something like F_v . Often F_v will be the completion of a number field F at a place v ; when this is not the case, v is just a decoration to remind the reader that F_v is a local field. We denote by G_{F_v} the absolute Galois group of F_v and by I_{F_v} the inertia subgroup.
- For a prime number p we denote by $\Sigma_{F,p}$ or Σ_p the set of places of F above p . We also write $\Sigma_{F,\infty}$ or Σ_∞ for the set of infinite places of F .
- We let E/\mathbb{Q}_p be an extension, typically finite, with ring of integers \mathcal{O} and residue field k . All our deformation theory takes place on the category of local \mathcal{O} -algebras.
- We use ρ (resp. $\bar{\rho}$) to denote p -adic (resp. mod p) representations of G_F . We typically denote representations of G_{F_v} by ρ_v or $\bar{\rho}_v$.
- We denote by χ_p the p -adic cyclotomic character and by $\bar{\chi}_p$ its reduction modulo p .
- The symbol Frob will always denote an arithmetic Frobenius.

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to me that eventually lead to the writing of this paper. Patrikis helped me immensely, answering several questions I had about Kisin's paper [Ki] and carefully reading and commenting on an earlier version of this paper.

2. LOCAL DEFORMATION RINGS

(2.1) Let F_v/\mathbb{Q}_p be a finite extension and let $\rho_v : G_{F_v} \rightarrow \mathrm{GL}_2(E)$ be a weight two representation, where E is any extension of \mathbb{Q}_p . Let V be the representation space of ρ .

- We say that ρ_v is *admissible of type A* if it is crystalline, ordinary and has cyclotomic determinant. This means that ρ_v is crystalline and there is an exact sequence

$$0 \rightarrow E(\chi_p \psi) \rightarrow V \rightarrow E(\psi^{-1}) \rightarrow 0$$

where ψ is an unramified character.

- We say that ρ_v is *admissible of type B* if it is crystalline, non-ordinary and has cyclotomic determinant.
- We say that ρ_v is *admissible of type C* if there is an exact sequence

$$0 \rightarrow E(\chi_p) \rightarrow V \rightarrow E \rightarrow 0.$$

Of course, this forces ρ_v to have cyclotomic determinant.

We say that ρ_v has *type ** if there is some finite extension F'_v/F_v such that $\rho_v|_{G_{F'_v}}$ is admissible of type *. As long as $\det \rho_v$ is a finite order character times the cyclotomic character ρ_v will have some type. It is possible for ρ_v to have both type A and C; we then say that ρ has type A/C.

Remark 2.1.1. Let f be a parallel weight two Hilbert eigenform over a totally real number field F , π the corresponding automorphic representation and ρ the corresponding p -adic Galois representation. Let v be a place of F over p . Then ρ has type A or B (resp. C) at v if and only if π_v is principal series or cuspidal (resp. special). The distinction between A and B amounts to a certain congruence for the Hecke eigenvalue. Note that $\rho|_{G_{F_v}}$ never has type A/C; in fact this is true for any global representation, *c.f.* Proposition 2.6.1.

(2.2) We now assume that F_v contains a non-trivial p th root of unity, so that the p -adic cyclotomic character χ_p of G_{F_v} reduces to the trivial character.

Proposition 2.2.1. *Let $\bar{\rho}_v : G_{F_v} \rightarrow \mathrm{GL}_2(k)$ be the trivial representation and let R^\square denotes its universal framed deformation ring. Let $*$ be A, B or C. There is then a unique quotient $R^{\square,*}$ of R^\square such that:*

- (1) $R^{\square,*} \otimes_{\mathcal{O}} \mathcal{O}_{E'}$ is a domain, for any finite extension E'/E .
- (2) $R^{\square,*}$ is flat over \mathcal{O} of relative dimension $[F_v : \mathbb{Q}_p] + 3$.
- (3) $R^{\square,*}[1/p]$ is formally smooth over E .
- (4) Let E'/E be a finite extension. A map $R^\square \rightarrow E'$ factors through $R^{\square,*}$ if and only if the corresponding representation is admissible of type *.

Proof. If $*$ is A or B then this is just [Ki, Corollary 2.5.16] (see also the proof of [Ki, Theorem 3.4.11]), with the improvement of [Ge] in the $*$ = B case. We give a proof in the case $*$ = C below. \square

(2.3) We now prove Proposition 2.2.1 for $*$ = C. We closely follow [Ki2, §2.4]. Let V_k be the trivial two dimensional representation of G_{F_v} . We denote by $\mathfrak{A}ug_{\mathcal{O}}$ the category of pairs (A, I) where A is an \mathcal{O} -algebra and $I \subset A$ is a nilpotent ideal with $\mathfrak{m}_{\mathcal{O}} A \subset I$. Let D^\times be the groupoid valued functor on $\mathfrak{A}ug_{\mathcal{O}}$ which assigns to (A, I) the category of pairs (V_A, ι_A) where:

- V_A is a free rank two A -module with a discrete action of G_{F_v} with determinant χ_p ; and
- $\iota_A : V_A \otimes_A A/I \rightarrow V_k \otimes_k A/I$ is an isomorphism of $(A/I)[G_{F_v}]$ -modules.

We define D^C to be the groupoid valued functor which assigns to (A, I) the category of triples (V_A, L_A, ι_A) where:

- V_A is a free rank two A -module with a discrete action of G_{F_v} with determinant χ_p ;
- $L_A \subset V_A$ is an A -line (that is, L_A is a rank one projective A -submodule for which V_A/L_A is projective) on which G_{F_v} acts by χ_p ; and
- $\iota_A : V_A \otimes_A A/I \rightarrow V_k \otimes_k A/I$ is an isomorphism of $(A/I)[G_{F_v}]$ -modules.

Note that D^C is analogous to the functor $D_{V_{\mathbb{F}}}^{\text{ord}, \chi}$ in [Ki2, §2.4]. We have the following analogue of [Ki2, Proposition 2.4.4].

Proposition 2.3.1. *We have the following:*

- (1) *There is a morphism of functors $D^C \rightarrow D^{\chi}$ taking (V_A, L_A, ι_A) to (V_A, ι_A) . This morphism is relatively representable and projective.*
- (2) *If R is a complete local ring with residue field k , ξ a map $\text{Spec}(R) \rightarrow D^{\chi}$ and \mathcal{L}_{ξ}^C the projective R -scheme representing $\text{Spec}(R) \times_{D^{\chi}} D^C$ then the morphism $\mathcal{L}_{\xi}^C \rightarrow \text{Spec}(R)$ becomes a closed immersion after inverting p .*
- (3) *If $\xi : \text{Spec}(R) \rightarrow D^{\chi}$ is formally smooth then \mathcal{L}_{ξ}^C is formally smooth over \mathcal{O} .*

Proof. The proof is the same as that given in [Ki2, Proposition 2.4.4], except for one point in (3). There, instead of using [Ki2, Lemma 2.4.2], one uses the fact that if $M \rightarrow M'$ is a surjection of \mathbb{Z}_p -modules on which p is nilpotent then $H^1(G_{F_v}, M(\chi_p)) \rightarrow H^1(G_{F_v}, M'(\chi_p))$ is surjective (this follows easily from Kummer theory). \square

Let $D^{\square, \chi}$ and $D^{\square, C}$ be the framed versions of D^{χ} and D^C . Then $D^{\square, \chi}$ is pro-representable and $D^{\square, \chi} \rightarrow D^{\chi}$ is formally smooth. Let $R^{\square, \chi}$ denote the complete local \mathcal{O} -algebra representing $D^{\square, \chi}$. The above proposition tells us that $\mathcal{L}^{\square, C} = R^{\square, \chi} \times_{D^{\chi}} D^C$ is a projective scheme over $\text{Spec}(R^{\square, \chi})$, is formally smooth over \mathcal{O} and the map $\mathcal{L}^{\square, C} \rightarrow \text{Spec}(R^{\square, \chi})$ becomes a closed embedding after inverting p . We define $\text{Spec}(R^{\square, C})$ to be the scheme theoretic image of $\mathcal{L}^{\square, C} \rightarrow \text{Spec}(R^{\square, \chi})$.

We now verify that $R^{\square, C}$ satisfies (1)–(4) of Proposition 2.2.1. The fiber of $\mathcal{L}^{\square, C}$ over the closed point of $R^{\square, C}$ is the projective line and so the proof of [Ki, Corollary 2.4.10] implies that $\mathcal{L}^{\square, C}$ is connected. As the image of a connected smooth scheme is a connected integral scheme, the ring $R^{\square, C}$ is a domain. Since the same argument works when considering deformations to $\mathcal{O}_{E'}$ algebras and that functor is represented by $R^{\square, C} \otimes_{\mathcal{O}} \mathcal{O}_{E'}$, we conclude that (1) holds. For (3) note that $\mathcal{L}^{\square, C} \rightarrow \text{Spec}(R^{\square, C})$ is an isomorphism after inverting p and the first space is smooth after inverting p . The proof of (4) goes just like the proof in [Ki2, Corollary 2.4.5].

We now verify (2). First note that there exists a ring homomorphism $R^{\square, C} \rightarrow \mathcal{O}$, as there exists a type C deformations of V_k to \mathcal{O} (e.g., $\chi_p \oplus 1$). This shows that $p \neq 0$ in $R^{\square, C}$. Flatness over \mathcal{O} now follows from (1). We now compute the dimension of $R^{\square, C}$. The relative dimension of $R^{\square, C}$ over \mathcal{O} is the same as the dimension of the tangent space of $\mathcal{L}^{\square, C}[1/p]$ at any point. Let ξ be the E -point of $\mathcal{L}^{\square, C}[1/p]$ corresponding to the representation $\chi_p \oplus 1$. The dimension of the tangent space at ξ is 2 (the contribution of the framing) plus the dimension of $H^1(G_{F_v}, E(\chi_p))$ (the contribution of deforming the representation), which gives a total of $[F_v : \mathbb{Q}_p] + 3$.

(2.4) Let F_v/\mathbb{Q}_{ℓ} be a finite extension with $\ell \neq p$ and let $\rho_v : G_{F_v} \rightarrow \text{GL}_2(E)$ be a representation with E any extension of \mathbb{Q}_p . Let V be the representation space of ρ_v .

- We say that ρ_v is *admissible of type AB* if it is unramified and has determinant χ_p .
- We say that ρ_v is *admissible of type C* if there is an exact sequence

$$0 \rightarrow E(\chi_p) \rightarrow V \rightarrow E \rightarrow 0.$$

Of course, this forces ρ_v to have determinant χ_p .

We say that ρ_v has *type ** if it is admissible of type $*$ over some finite extension F'_v/F_v . As long as $\det \rho_v$ is a finite order character times the cyclotomic character, ρ_v will have type AB or C . As in the previous case, it is possible for ρ_v to have type AB and C ; we then say that ρ_v has type AB/C .

(2.5) We will need some results on deformations of given types, as in the case where $v \mid p$. We assume that F_v contains the p th roots of unity.

Proposition 2.5.1. *Let $\bar{\rho}_v : G_{F_v} \rightarrow \text{GL}_2(k)$ be the trivial representation and let R^{\square} denote its universal framed deformation ring. Let $*$ be one of AB or C . There is then a unique quotient $R^{\square, *}$ of R^{\square} satisfying the following properties:*

- (1) $R^{\square, *} \otimes_{\mathcal{O}} \mathcal{O}_{E'}$ is a domain, for any finite E'/E .
- (2) $R^{\square, *}$ is flat over \mathcal{O} of relative dimension 3.
- (3) $R^{\square, *}[1/p]$ is formally smooth over E .

- (4) Let E'/E be a finite extension. A map $R^\square \rightarrow E'$ factors through $R^{\square,*}$ if and only if the corresponding representation is admissible of type $*$.

Proof. For $* = AB$ this is easy; for $* = C$ it is [Ki, Corollary 2.6.7] with $\gamma = 1$. \square

(2.6) We now introduce some terminology for dealing with types that we will use for the rest of the paper. By a p -type we mean one of the symbols A, B, C or A/C . By a *prime-to- p -type* we mean one of the symbols AB, C or AB/C . The types A/C and AB/C are *indefinite*; the others are *definite*. For two types X and Y we write $X \approx Y$ if X and Y are equal or if (X, Y) is one of $(A, A/C), (C, A/C), (AB, AB/C)$ or $(C, AB/C)$ up to switching order. For example, $A \approx A/C$ but $B \not\approx A/C$. Note that if X and Y are definite types then $X \approx Y$ is equivalent to $X = Y$.

Let F_v/\mathbb{Q}_p be a finite extension and $\rho_v : G_{F_v} \rightarrow \mathrm{GL}_2(E)$ a weight two representation. We define t_{ρ_v} to be the appropriate p -type, e.g., $t_{\rho_v} = A/C$ if ρ_v has type A and type C while $t_{\rho_v} = A$ if ρ_v has type A and not type C . We similarly define t_{ρ_v} when F_v is a finite extension of \mathbb{Q}_ℓ with $\ell \neq p$. We use the phrase “ ρ_v has type $*$ ” to mean $t_{\rho_v} \approx *$ (which is how the phrase has been defined in the previous sections). For a definite type $*$, we use the phrase “ ρ_v has definite type $*$ ” to mean $t_{\rho_v} = *$.

Let F be a number field and S a finite set of primes of F . By a *type function* over F on S we mean a function t on S such that $t(v)$ is a p -type (resp. prime-to- p -type) for v above p (resp. not above p). We call t *definite* if $t(v)$ is definite for all $v \in S$. If t and t' are two type functions over F on S we write $t \approx t'$ if $t(v) \approx t'(v)$ for all $v \in S$. For a type function t over F on S and an extension F'/F we let $t|_{F'}$ be the type function over F' on the primes over S which assigns to a place v' the value $t(v)$ where v is the place of F over which v' lies.

For a weight two representation $\rho : G_{F,S} \rightarrow \mathrm{GL}_2(E)$, with E an extension of \mathbb{Q}_p , we let t_ρ be the type function on S which assigns to v the type $t_{\rho|_{G_{F_v}}}$. We call t_ρ the *type* of ρ . We have the following result:

Proposition 2.6.1. *Let $\rho : G_F \rightarrow \mathrm{GL}_2(\overline{\mathbb{Q}_p})$ be an irreducible, odd, weight two representation which is potentially modular. Then t_ρ is definite.*

Proof. Assume that $v \nmid p$ is a finite place for which $t_\rho(v)$ is not definite. We can then find a finite totally real extension F'/F and a place w of F' over v such that (1) $\rho|_{G_{F'}} = \rho_f$ for a cuspidal Hilbert eigenform f over F' ; and (2) $\rho|_{G_{F'_w}}$ is unramified and an extension of 1 by χ_p . Statement (2) shows that $\mathrm{tr} \rho(\mathrm{Frob}_w) = \mathbf{N}w + 1$, which contradicts the Ramanujan conjecture for f (see [Bl]). Thus no such v exists. To show that $t_\rho(v)$ is definite for $v \mid p$ one can apply a similar argument to crystalline Frobenius. \square

3. AN $R = \mathbb{T}$ THEOREM

(3.1) Let F be a totally real number field. Fix a representation

$$\bar{\rho} : G_F \rightarrow \mathrm{GL}_2(k).$$

(Recall that k is the residue field of E , a finite extension of \mathbb{Q}_p .) We assume the following:

- (A1) The representation $\bar{\rho}|_{G_{F(\zeta_p)}}$ is absolutely irreducible.
- (A2) If $p = 5$ and the projective image of $\bar{\rho}$ is $\mathrm{PGL}_2(\mathbb{F}_5)$ then $[F(\zeta_p) : F] = 4$.
- (A3) The representation $\bar{\rho}$ is odd.
- (A4) The representation $\bar{\rho}$ is everywhere unramified.
- (A5) We have $\det \bar{\rho} = \overline{\chi}_p$.
- (A6) The field F has even degree over \mathbb{Q} .
- (A7) The field k contains the eigenvalues of the image of $\bar{\rho}$.

Hypothesis (A6) is used in §3.6 to ensure the existence of a certain quaternion algebra. Hypothesis (A7) is mild: we can always replace k by its quadratic extension to ensure that it holds.

(3.2) We define a *deformation datum* to be a tuple $\mathcal{D}^\circ = (t, \Sigma^{\mathrm{ram}}, \Sigma^{\mathrm{aux}})$ where:

- Σ^{ram} is a finite set of primes disjoint from Σ_p .
- t is a definite type function over F on $\Sigma_p \cup \Sigma^{\mathrm{ram}}$.
- Σ^{aux} is a finite set of primes disjoint from $\Sigma_p \cup \Sigma^{\mathrm{ram}}$ such that each $v \in \Sigma^{\mathrm{aux}}$ satisfies $\mathbf{N}v \not\equiv 1 \pmod{p}$ and $\mathrm{tr} \bar{\rho}(\mathrm{Frob}_v) \neq \pm(1 + \mathbf{N}v)$.

We put $\Sigma_* = t^{-1}(*)$, $\Sigma_{p,*} = \Sigma_p \cap \Sigma_*$ and $\Sigma_*^{\text{ram}} = \Sigma^{\text{ram}} \cap \Sigma_*$. Of particular importance will be the set $\Sigma_C = \Sigma_{p,C} \cup \Sigma_C^{\text{ram}}$.

We fix for the rest of §3 a deformation datum \mathcal{D}° . We impose the following hypotheses:

- (A8) The representation $\bar{\rho}$ is trivial at all places above $\Sigma_p \cup \Sigma^{\text{ram}}$.
- (A9) The set Σ_C has even cardinality.

The condition (A9) is used to ensure the existence of the quaternion algebra of §3.6.

We define a *TW-extension* (of the datum \mathcal{D}°) to be a tuple $\mathcal{D} = (\Sigma^{\text{TW}}, \{\alpha_v\})$ where:

- Σ^{TW} is a finite set of primes disjoint from $\Sigma_p \cup \Sigma^{\text{ram}} \cup \Sigma^{\text{aux}}$ such that for each $v \in \Sigma^{\text{aux}}$ we have $\mathbf{N} v = 1 \pmod{p}$ and the eigenvalues of $\bar{\rho}(\text{Frob}_v)$ are distinct and belong to k .
- α_v is an eigenvalue of $\bar{\rho}(\text{Frob}_v)$, for $v \in \Sigma^{\text{TW}}$.

The *trivial TW-extension* is the one where $\Sigma^{\text{TW}} = \emptyset$. We often write \mathcal{D}° for the trivial TW-extension of \mathcal{D}° .

Remark 3.2.1. We are eventually trying to prove that certain lifts ρ of $\bar{\rho}$ are modular, given that $\bar{\rho}$ is modular. In such applications, $\Sigma^{\text{ram}} = \Sigma_C^{\text{ram}}$ will be the set of primes away from p at which ρ ramifies, Σ^{aux} will be a set containing a single prime (used to ensure that the subgroup U° of §3.7 satisfies the condition $(*)$ of §3.6) and Σ^{TW} will take on varying values (these are the primes used in the Taylor-Wiles method).

(3.3) Let \mathcal{D} be a TW-extension of \mathcal{D}° . We define several local framed deformation rings:

- We let R_v^\square be the universal framed deformation ring of $\bar{\rho}|_{G_{F_v}}$.
- For $v \in \Sigma_p \cup \Sigma^{\text{ram}}$ we let $R_{\mathcal{D},v}^\square$ be the “admissible of type $t(v)$ ” quotient of R_v^\square given by Proposition 2.2.1 in case $v \in \Sigma_p$ Proposition 2.5.1 in case $v \in \Sigma^{\text{ram}}$.
- For $v \in \Sigma_{p,A} \cup \Sigma_{p,B}$ we also have a ring $\tilde{R}_{\mathcal{D},v}^\square$ (see [Ki, §3.4.7]). This ring enjoys the ring theoretic properties listed in Proposition 2.2.1 and admits a finite map from $R_{\mathcal{D},v}^\square$ which becomes an isomorphism after inverting p ; other than these facts, the details of $\tilde{R}_{\mathcal{D},v}^\square$ will not be important to us.

We also need some semi-local framed deformation rings. In what follows, “tensor product” means the completed tensor product.

- We let $R^{\square, \text{loc}}$ be the tensor product of the rings R_v^\square for $v \in \Sigma_p \cup \Sigma^{\text{ram}}$.
- We let $R_{\mathcal{D}}^{\square, \text{loc}}$ be the tensor product of the rings $R_{\mathcal{D},v}^\square$ for $v \in \Sigma_p \cup \Sigma^{\text{ram}}$.
- We let $\tilde{R}_{\mathcal{D}}^{\square, \text{loc}}$ be the tensor product of the $\tilde{R}_{\mathcal{D},v}^\square$ for $v \in \Sigma_{p,A} \cup \Sigma_{p,B}$ together with the $R_{\mathcal{D},v}^\square$ for $v \in \Sigma_{p,C} \cup \Sigma^{\text{ram}}$.

The above rings depend only on \mathcal{D}° and not the TW-extension \mathcal{D} . Let $\Sigma(\mathcal{D}) = \Sigma_p \cup \Sigma^{\text{ram}} \cup \Sigma^{\text{aux}} \cup \Sigma^{\text{TW}}$. We now define some global framed deformation rings.

- Let $R_{\Sigma(\mathcal{D})}^\square$ be the universal ring classifying deformations of $\bar{\rho}$ unramified outside of $\Sigma(\mathcal{D})$, with determinant χ_p and with framings at each place in $\Sigma_p \cup \Sigma^{\text{ram}}$.
- Let $R_{\mathcal{D}}^\square$ be the tensor product $R_{\mathcal{D}}^{\square, \text{loc}} \otimes_{R^{\square, \text{loc}}} R_{\Sigma(\mathcal{D})}^\square$.
- Let $\tilde{R}_{\mathcal{D}}^\square$ be the tensor product $\tilde{R}_{\mathcal{D}}^{\square, \text{loc}} \otimes_{R^{\square, \text{loc}}} R_{\Sigma(\mathcal{D})}^\square = \tilde{R}_{\mathcal{D}}^{\square, \text{loc}} \otimes_{R_{\mathcal{D}}^{\square, \text{loc}}} R_{\mathcal{D}}^\square$.

We will make use of a few unframed deformation rings as well. These exist since $\bar{\rho}$ is absolutely irreducible.

- We let $R_{\Sigma(\mathcal{D})}$ be the universal ring classifying deformations of $\bar{\rho}$ unramified outside of $\Sigma(\mathcal{D})$ with determinant χ_p .
- We let $R_{\mathcal{D}}$ be the quotient of $R_{\Sigma(\mathcal{D})}$ analogous to the quotient $R_{\mathcal{D}}^\square$ of $R_{\Sigma(\mathcal{D})}^\square$.
- The ring $\tilde{R}_{\mathcal{D}}$ is defined similarly.

None of these rings depend on the choice of eigenvalues α_v in the TW-extension. These will matter later, though.

Remark 3.3.1. Let \mathcal{D}_1° be a deformation datum and let \mathcal{D}_2° be the deformation datum gotten by deleting Σ_{AB}^{ram} from Σ^{ram} . Then $R_{\mathcal{D}_1^\circ} = R_{\mathcal{D}_2^\circ}$, as $R_{\Sigma(\mathcal{D}_1^\circ)}$ allows ramification at Σ_{AB}^{ram} but the local condition imposed by $R_{\mathcal{D}_1^\circ, v}^\square$ for $v \in \Sigma_{AB}^{\text{ram}}$ exactly removes this allowance. The equality $R_{\mathcal{D}_1^\circ}^\square = R_{\mathcal{D}_2^\circ}^\square$ is not true, however, as the first ring has framings at Σ_{AB}^{ram} while the second does not.

(3.4) Define

$$\begin{aligned} h &= h(\mathcal{D}^\circ) = \dim_k H^1(G_{F, \Sigma(\mathcal{D}^\circ)}, \text{ad}^\circ \bar{\rho}(1)) \\ g &= g(\mathcal{D}^\circ) = h - [F : \mathbb{Q}] + \#\Sigma_p + \#\Sigma^{\text{ram}} - 1. \end{aligned}$$

The purpose of this section is to recall the following result.

Proposition 3.4.1. *Given an integer n we can find a set of primes Σ^{TW} of F satisfying the following conditions:*

- (1) *The set Σ^{TW} is disjoint from $\Sigma(\mathcal{D}^\circ)$.*
- (2) *We have $\mathbf{N}v = 1 \pmod{p^n}$ for each $v \in \Sigma^{\text{TW}}$.*
- (3) *We have $\#\Sigma^{\text{TW}} = h$.*
- (4) *The eigenvalues of $\bar{\rho}(\text{Frob}_v)$ are distinct for each $v \in \Sigma^{\text{TW}}$.*
- (5) *For each $v \in \Sigma^{\text{TW}}$ choose an eigenvalue α_v of $\bar{\rho}(\text{Frob}_v)$ and let $\mathcal{D} = (\Sigma^{\text{TW}}, \{\alpha_v\})$ be the corresponding TW-extension of \mathcal{D}° . Then $R_{\mathcal{D}}^\square$ is topologically generated over $R_{\mathcal{D}}^{\square, \text{loc}}$ by g elements.*

Proof. Exactly as in [Ki, Proposition 3.2.5]. The proof uses (A1), (A2) and (A3) as well as the assumptions on the primes in Σ^{aux} . \square

(3.5) Let \mathcal{D} be a TW-extension of \mathcal{D}° . Define $\Delta_{\mathcal{D}}$ to be the maximal p -power quotient of $\prod_{v \in \Sigma^{\text{TW}}} (\mathcal{O}_{F_v} / \mathfrak{p}_v)^\times$. We denote by $\mathfrak{a}_{\mathcal{D}}$ the augmentation ideal of $\mathcal{O}[\Delta_{\mathcal{D}}]$.

We now give $R_{\Sigma(\mathcal{D})}$ the structure of an $\mathcal{O}[\Delta_{\mathcal{D}}]$ -algebra. Let v be an element of Σ^{TW} . The restriction of the universal representation $G_F \rightarrow \text{GL}_2(R_{\Sigma(\mathcal{D})})$ to the decomposition group G_{F_v} is a sum of two characters $\eta_1 \oplus \eta_2$, where $\eta_i : G_{F_v} \rightarrow (R_{\Sigma(\mathcal{D})})^\times$ (see [DDT, Lemma 2.44]). Reducing η_i modulo the maximal ideal of $R_{\Sigma(\mathcal{D})}$ gives an unramified character of G_{F_v} with values in k^\times whose value on Frob_v is one of the two eigenvalues of $\bar{\rho}(\text{Frob}_v)$. Change the labeling if necessary so that η_1 corresponds to the chosen eigenvalue α_v . By class field theory (normalized so that uniformizers correspond to arithmetic Frobenii), η_1 gives a map $F_v^\times \rightarrow (R_{\Sigma(\mathcal{D})})^\times$ whose restriction to $\mathcal{O}_{F_v}^\times$ factors through the maximal p -power quotient of $(\mathcal{O}_{F_v} / \mathfrak{p}_v)^\times$. Taking the product of these over $v \in \Sigma^{\text{TW}}$ gives a group homomorphism $\Delta_{\mathcal{D}} \rightarrow (R_{\Sigma(\mathcal{D})})^\times$, which gives $R_{\Sigma(\mathcal{D})}$ the structure of an $\mathcal{O}[\Delta_{\mathcal{D}}]$ -algebra. Note that this gives any $R_{\Sigma(\mathcal{D})}$ -algebra (e.g., $R_{\Sigma(\mathcal{D})}^\square$ or $R_{\mathcal{D}}^\square$) the structure of an $\mathcal{O}[\Delta_{\mathcal{D}}]$ -algebra.

Proposition 3.5.1. *The natural map $R_{\mathcal{D}}^\square \rightarrow R_{\mathcal{D}^\circ}^\square$ is surjective with kernel $\mathfrak{a}_{\mathcal{D}} R_{\mathcal{D}}^\square$. The same statement holds for the tilde rings.*

Proof. The map on functors $\text{Hom}(R_{\mathcal{D}^\circ}^\square, -) \rightarrow \text{Hom}(R_{\mathcal{D}}^\square, -)$ is obviously injective, and so the map on tangent spaces is injective, and so the map the other way on cotangent spaces is surjective. Nakayama's lemma now gives that $R_{\mathcal{D}}^\square \rightarrow R_{\mathcal{D}^\circ}^\square$ is surjective. One easily sees that this map contains $\mathfrak{a}_{\mathcal{D}} R_{\mathcal{D}}^\square$ in its kernel and that the representation $G_F \rightarrow \text{GL}_2(R_{\mathcal{D}}^\square / \mathfrak{a}_{\mathcal{D}} R_{\mathcal{D}}^\square)$ is unramified at Σ^{TW} . It follows that there is a map $R_{\mathcal{D}^\circ} \rightarrow R_{\mathcal{D}}^\square / \mathfrak{a}_{\mathcal{D}} R_{\mathcal{D}}^\square$ and this is easily seen to be the inverse of the natural map in the other direction. \square

(3.6) Let D be the unique quaternion algebra over F ramified at exactly the infinite places and at Σ_C . This exists by (A6) and (A9). Pick a maximal order \mathcal{O}_D of D and for each finite place $v \notin \Sigma_C$ pick an isomorphism $\mathcal{O}_{D_v} \rightarrow M_2(\mathcal{O}_{F_v})$. Let $U = \prod U_v$ be a compact open subgroup of $(D \otimes_F \mathbb{A}_F^f)^\times$, where each U_v is a compact open subgroup of $\mathcal{O}_{D_v}^\times$. We call such a compact open subgroup *standard*. We let $S_2(U)$ denote the set of functions

$$f : D^\times \backslash (D \otimes_F \mathbb{A}_F^f)^\times / ((\mathbb{A}_F^f)^\times \cdot U) \rightarrow \mathcal{O}.$$

If v is a place at which U is maximal and D is unramified then the Hecke operators T_v acts on $S_2(U)$. We let $\mathbb{T}(U)$ (resp. $\mathbb{T}^{(p)}(U)$) be the \mathcal{O} -subalgebra of $\text{End}(S_2(U))$ generated by the T_v for all such v (resp. for all such v not above p).

We will often need to impose the following condition on our subgroup U :

$$\text{For all } t \in (D \otimes_F \mathbb{A}_F^f)^\times \text{ the group } ((\mathbb{A}_F^f)^\times \cdot U \cap t^{-1} D^\times t) / F^\times \text{ has prime-to-} p \text{ order.} \quad (*)$$

This condition is equivalent to the following one:

$$\text{The stabilizers of } (\mathbb{A}_F^f)^\times \cdot U \text{ acting on } D^\times \backslash (D \otimes_F \mathbb{A}_F^f)^\times \text{ have prime-to-} p \text{ order.}$$

Obviously, if this condition holds for U then it holds for any subgroup of U .

(3.7) We now make the following hypothesis on the representation $\bar{\rho}$. We assume that there exists a standard compact open subgroup U° of $(D \otimes_F \mathbb{A}_F^f)^\times$ and a maximal ideal \mathfrak{m} of $\mathbb{T}(U^\circ)$ satisfying the following conditions:

- (B1) The group U_v° is maximal except for $v \in \Sigma^{\text{aux}}$.
- (B2) The group U° satisfies the condition $(*)$.
- (B3) For $v \notin \Sigma(\mathcal{D}^\circ)$ the image of T_v in $\mathbb{T}(U^\circ)/\mathfrak{m}$ is equal to $\text{tr } \bar{\rho}(\text{Frob}_v)$.
- (B4) For $v \in \Sigma_{p,A}$ we have $T_v \notin \mathfrak{m}$.
- (B5) For $v \in \Sigma_{p,B}$ we have $T_v \in \mathfrak{m}$.
- (B6) The residue field of \mathfrak{m} is k .

Condition (B6) is mild and can always be ensured by enlarging k . The ideal \mathfrak{m} is forced to be non-Eisenstein since the representation $\bar{\rho}$ is absolutely irreducible.

Remark 3.7.1. Let U be a standard compact open with U_v maximal for some place $v \in \Sigma_p \setminus \Sigma_{p,C}$. Let f be an element of $S_2(U)$ which is an eigenform for $\mathbb{T}(U)$. Associated to f is a maximal ideal \mathfrak{m} of $\mathbb{T}(U)$. One then knows that the p -adic Galois representation ρ_f associated to f is admissible of type A (resp. B) at v if and only if $T_v \notin \mathfrak{m}$ (resp. $T_v \in \mathfrak{m}$). This follows from the second part of [Ki, Lemma 3.4.2]. This is the reason for conditions (B4) and (B5) above.

(3.8) Let \mathcal{D} be a TW-extension of \mathcal{D}° . Define standard compact open subgroups $U_{\mathcal{D}}$ and $U_{\mathcal{D}}^-$ by $(U_{\mathcal{D}})_v = (U_{\mathcal{D}}^-)_v = U_v^\circ$ for $v \notin \Sigma^{\text{TW}}$ and

$$(U_{\mathcal{D}}^-)_v = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{GL}_2(\mathcal{O}_{F_v}) \mid c \in \mathfrak{p}_v \right\}$$

and

$$(U_{\mathcal{D}})_v = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{GL}_2(\mathcal{O}_{F_v}) \mid c \in \mathfrak{p}_v \text{ and } ad^{-1} \text{ maps to } 1 \text{ in } \Delta_{\mathcal{D}} \right\}$$

for $v \in \Sigma^{\text{TW}}$. We identify $U_{\mathcal{D}}^-/U_{\mathcal{D}}$ with the group $\Delta_{\mathcal{D}}$ by locally mapping a matrix to ad^{-1} . As such, the group $\Delta_{\mathcal{D}}$ naturally acts on the space $S_2(U_{\mathcal{D}})$.

For any four of the Hecke algebras $\mathbb{T}(U_{\mathcal{D}})$, $\mathbb{T}^{(p)}(U_{\mathcal{D}})$, $\mathbb{T}(U_{\mathcal{D}}^-)$ or $\mathbb{T}^{(p)}(U_{\mathcal{D}}^-)$ we put a $+$ in the superscript to indicate the Hecke algebra generated by the given algebra and the Atkin-Lehner operators U_v for $v \in \Sigma^{\text{TW}}$. We regard \mathfrak{m} as an ideal of each of these algebras.

Proposition 3.8.1. *For each $v \in \Sigma^{\text{TW}}$ choose an eigenvalue β_v of $\bar{\rho}(\text{Frob}_v)$. Then the ideal of $\mathbb{T}^+(U_{\mathcal{D}}^-)$ generated by \mathfrak{m} and the $U_v - \beta_v$ for $v \in \Sigma^{\text{TW}}$ is a maximal ideal. Every maximal ideal above \mathfrak{m} is of this form and no two are the same.*

Proof. This follows from statement (2) of [Tay, Lemma 1.6] and statement (1) of [Tay, Corollary 1.8] \square

We let $\mathfrak{m}_{\mathcal{D}}^-$ be the maximal ideal of $\mathbb{T}^+(U_{\mathcal{D}}^-)$ corresponding to the eigenvalues $\{\alpha_v\}$ given in the datum \mathcal{D} and let $\mathfrak{m}_{\mathcal{D}}$ be the induced maximal ideal of $\mathbb{T}^+(U_{\mathcal{D}})$. We then define $\mathbb{T}_{\mathcal{D}}$ (resp. $\mathbb{T}_{\mathcal{D}}^{(p)}$) to be the localization $\mathbb{T}^+(U_{\mathcal{D}})_{\mathfrak{m}_{\mathcal{D}}}$ (resp. $\mathbb{T}^{(p),+}(U_{\mathcal{D}})_{\mathfrak{m}_{\mathcal{D}}}$). We also put $M_{\mathcal{D}} = S_2(U_{\mathcal{D}})_{\mathfrak{m}_{\mathcal{D}}}$, which is a module over $\mathbb{T}_{\mathcal{D}}$. Note that under our convention of identifying \mathcal{D}° with its trivial TW-extension we have $\mathbb{T}_{\mathcal{D}^\circ} = \mathbb{T}(U^\circ)_{\mathfrak{m}}$, and similarly for the prime-to- p version.

Proposition 3.8.2. *The space $M_{\mathcal{D}}$ is a free $\mathcal{O}[\Delta_{\mathcal{D}}]$ -module and there is a natural isomorphism $M_{\mathcal{D}}/\mathfrak{a}_{\mathcal{D}} M_{\mathcal{D}} \rightarrow M_{\mathcal{D}^\circ}$.*

Proof. This is proved as [Tay, Corollary 2.4]. \square

(3.9) Fix a TW-extension \mathcal{D} of \mathcal{D}° . We then have the following:

Proposition 3.9.1. *There is a representation $\rho : G_F \rightarrow \text{GL}_2(\mathbb{T}_{\mathcal{D}}^{(p)})$ which is unramified outside of $\Sigma(\mathcal{D})$, has determinant χ_p and satisfies $\text{tr } \rho(\text{Frob}_v) = T_v$ for $v \notin \Sigma(\mathcal{D})$. The corresponding map $R_{\Sigma(\mathcal{D})} \rightarrow \mathbb{T}_{\mathcal{D}}^{(p)}$ is surjective.*

Proof. This follows from [Tay2] and the Jacquet-Langlands correspondence. \square

We put $\mathbb{T}_{\mathcal{D}}^\square = R_{\Sigma(\mathcal{D})}^\square \otimes_{R_{\Sigma(\mathcal{D})}} \mathbb{T}_{\mathcal{D}}$ and similarly for the prime-to- p version. We also define $M_{\mathcal{D}}^\square = \mathbb{T}_{\mathcal{D}}^\square \otimes_{\mathbb{T}_{\mathcal{D}}} M_{\mathcal{D}}$. We regard $M_{\mathcal{D}}$ as an $R_{\Sigma(\mathcal{D})}$ -module via the map $R_{\Sigma(\mathcal{D})} \rightarrow \mathbb{T}_{\mathcal{D}}^{(p)}$, and similarly for the boxed version. We note that the two actions of $\Delta_{\mathcal{D}}$ on $M_{\mathcal{D}}$ — one coming from the $R_{\Sigma(\mathcal{D})}$ -module structure, the other from the identification $U_{\mathcal{D}}^-/U_{\mathcal{D}} = \Delta_{\mathcal{D}}$ — agree (this follows from [Tay, Corollary 1.8]).

Proposition 3.9.2. *The composite map*

$$R^{\square, \text{loc}} \rightarrow R_{\Sigma(\mathcal{D})}^{\square} \rightarrow \mathbb{T}_{\mathcal{D}}^{\square, (p)} \rightarrow \mathbb{T}_{\mathcal{D}}^{\square}$$

factors through $R_{\mathcal{D}}^{\square, \text{loc}}$. The resulting map $R_{\mathcal{D}}^{\square} \rightarrow \mathbb{T}_{\mathcal{D}}^{\square}$ extends naturally to a surjection

$$\tilde{R}_{\mathcal{D}}^{\square} \rightarrow \mathbb{T}_{\mathcal{D}}^{\square}.$$

Proof. For the first statement, it suffices to show that for each $v \in \Sigma_p \cup \Sigma^{\text{ram}}$ the map $R_v^{\square} \rightarrow \mathbb{T}_{\mathcal{D}}^{\square}$ factors through $R_{\mathcal{D}, v}^{\square}$. For $v \in \Sigma_{p, A} \cup \Sigma_{p, B} \cup \Sigma_C^{\text{ram}}$ this is shown in [Ki, Lemma 3.4.9]. The proof for $v \in \Sigma_{AB}^{\text{ram}}$ is the same as the $v \in \Sigma_C^{\text{ram}}$ case. We must handle the $v \in \Sigma_{p, C}$ case. As in [Ki, Lemma 3.4.9] it suffices to show that for any map $\mathbb{T}_{\mathcal{D}}^{\square} \rightarrow E'$, with E' an extension of E , the resulting map $R_v^{\square} \rightarrow E'$ factors through $R_{\mathcal{D}, v}^{\square}$. The map $\mathbb{T}_{\mathcal{D}}^{\square} \rightarrow E'$ determines a Hilbert modular form f which is special of conductor one at v . We thus have

$$\rho_f|_{G_{F_v}} = \begin{pmatrix} \chi_p \eta & * \\ & \eta \end{pmatrix}$$

where η is unramified and $\eta^2 = 1$. Since $\rho_f|_{G_{F_v}}$ reduces to the trivial representation (and $p \neq 2$) we conclude $\eta = 1$. Thus ρ_f is admissible of type C at v . This proves that $R_v^{\square} \rightarrow E'$ factors through $R_{\mathcal{D}, v}^{\square}$. That $R_{\mathcal{D}}^{\square} \rightarrow \mathbb{T}_{\mathcal{D}}^{\square}$ extends to a surjection from $\tilde{R}_{\mathcal{D}}^{\square}$ is a local statement at the primes in $\Sigma_{p, A} \cup \Sigma_{p, B}$ and is proved in [Ki, Lemma 3.4.9]. \square

(3.10) We recall the following result from [Ki, Proposition 3.3.1].

Proposition 3.10.1. *Let B be a complete, local, flat \mathcal{O} -algebra, which is a domain of dimension $d + 1$ and such that $B[1/p]$ is formally smooth over E . Let R be a local B -algebra and M a non-zero R -module. Suppose that there are integers h and j such that for any integer n we can find:*

- *a local B -algebra R' , topologically generated over B by $h + j - d$ elements;*
- *a map $\mathcal{O}[[x_1, \dots, x_h, y_1, \dots, y_j]] \rightarrow R'$;*
- *a surjection of B -algebras $R' \rightarrow R$ with kernel $(x_1, \dots, x_h)R'$;*
- *an R' -module M' ; and*
- *a surjection of R' -modules $M' \rightarrow M$ with kernel $(x_1, \dots, x_h)M'$,*

such that the following condition holds:

- *if $\mathfrak{b}' \subset \mathcal{O}[[x_1, \dots, x_h, y_1, \dots, y_j]]$ is the annihilator of M' then*

$$\mathfrak{b}' \subset ((1 + x_1)^{p^n} - 1, \dots, (1 + x_h)^{p^n} - 1)$$

and M' is finite free over $\mathcal{O}[[x_1, \dots, x_h, y_1, \dots, y_j]]/\mathfrak{b}'$.

Then R is a finite $\mathcal{O}[[y_1, \dots, y_j]]$ -algebra and $M[1/p]$ is a finite, projective and faithful $R[1/p]$ -module.

(3.11) The aim of this section is to prove the following theorem. We follow [Ki, Theorem 3.4.11].

Theorem 3.11.1. *The map $\tilde{R}_{\mathcal{D}^\circ}^{\square} \rightarrow \mathbb{T}_{\mathcal{D}^\circ}^{\square}$ is surjective and has p -power torsion kernel. The ring $R_{\mathcal{D}^\circ}$ is finite over \mathcal{O} .*

Proof. We apply Proposition 3.10.1 with:

$$B = \tilde{R}_{\mathcal{D}^\circ}^{\square, \text{loc}}, \quad R = \tilde{R}_{\mathcal{D}^\circ}^{\square}, \quad M = M_{\mathcal{D}^\circ}^{\square}, \quad h = h(\mathcal{D}^\circ), \quad j = 4(\#\Sigma_p + \#\Sigma^{\text{ram}}) - 1.$$

By Proposition 2.2.1, Proposition 2.5.1 and [Ki, Lemma 3.4.12] B is a complete, local, flat \mathcal{O} -algebra which is a domain and such that $B[1/p]$ is smooth over E . Its dimension is $d + 1$ where d is the relative dimension of $R_{\mathcal{D}}^{\square, \text{loc}}$ over \mathcal{O} . Thus

$$d = \sum_{v \in \Sigma_p} ([F_v : \mathbb{Q}_p] + 3) + \sum_{v \in \Sigma^{\text{ram}}} 3 = [F : \mathbb{Q}] + 3\#\Sigma_p + 3\#\Sigma^{\text{ram}}$$

Let n be a given integer, let \mathcal{D} be the TW-extension produced by Proposition 3.4.1 and put $R' = \tilde{R}_{\mathcal{D}}^{\square}$ and $M' = M_{\mathcal{D}}^{\square}$. The morphism $R_{\Sigma(\mathcal{D})}^{\square} \rightarrow R_{\Sigma(\mathcal{D})}^{\square}$ is formally smooth of relative dimension j , and so we can identify $R_{\Sigma(\mathcal{D})}^{\square}$ with $R_{\Sigma(\mathcal{D})}^{\square}[[y_1, \dots, y_j]]$. In particular, this gives R' the structure of an $\mathcal{O}[[y_1, \dots, y_j]]$ -algebra. The group $\Delta_{\mathcal{D}}$ is a quotient of the product of h cyclic groups. We can thus write $\mathcal{O}[\Delta_{\mathcal{D}}]$ as a quotient of $\mathcal{O}[[x_1, \dots, x_h]]$ in such a way that the images of the x_i generate the augmentation ideal $\mathfrak{a}_{\mathcal{D}}$. As R' is

an algebra over $\mathcal{O}[\Delta_{\mathcal{D}}]$ we can regard it as an algebra over $\mathcal{O}[[x_1, \dots, x_h]]$. We have thus defined a map $\mathcal{O}[[x_1, \dots, x_h, y_1, \dots, y_j]] \rightarrow R'$.

Proposition 3.5.1 shows that the natural map $R' \rightarrow R$ is surjective and has kernel $(x_1, \dots, x_h)R'$. Similarly, Proposition 3.8.2 shows that the natural map $M' \rightarrow M$ is surjective and has kernel $(x_1, \dots, x_h)M'$. That proposition, together with obvious properties of $\Delta_{\mathcal{D}}$, show that \mathfrak{b}' satisfies the necessary condition and that M' is finite and free over $\mathcal{O}[[x_1, \dots, x_h, y_1, \dots, y_j]]/\mathfrak{b}'$. Finally, we need to check that R' is topologically generated as a B -algebra by $h + j - d$ elements. From the manner in which we chose \mathcal{D} we know that R' is generated by $g = h - [F : \mathbb{Q}] + \#\Sigma_p + \#\Sigma^{\text{ram}} - 1$ elements over B . It suffices, therefore, to show $g \leq h + j - d$. In fact,

$$\begin{aligned} h + j - d &= h + 4(\#\Sigma_p + \#\Sigma^{\text{ram}}) - 1 - [F : \mathbb{Q}] - 3\#\Sigma_p - 3\#\Sigma^{\text{ram}} \\ &= h - [F : \mathbb{Q}] + \#\Sigma_p + \#\Sigma^{\text{ram}} - 1 \\ &= g. \end{aligned}$$

We have thus verified the hypotheses of Proposition 3.10.1. It follows that $M_{\mathcal{D}^\circ}^\square[1/p]$ is a faithful $\tilde{R}_{\mathcal{D}^\circ}^\square[1/p]$ -module. Since the $\tilde{R}_{\mathcal{D}^\circ}^\square$ -module structure on $M_{\mathcal{D}^\circ}^\square$ comes via the map $\text{map } \tilde{R}_{\mathcal{D}^\circ}^\square \rightarrow \mathbb{T}_{\mathcal{D}^\circ}^\square$, it follows that this map must have p -power torsion kernel, proving the first statement of the theorem. Proposition 3.10.1 also implies that $\tilde{R}_{\mathcal{D}^\circ}^\square$ is finite over $\mathcal{O}[[y_1, \dots, y_j]]$. Since $\tilde{R}_{\mathcal{D}^\circ}^\square$ is identified with $\tilde{R}_{\mathcal{D}^\circ}[y_1, \dots, y_j]$ it follows that $\tilde{R}_{\mathcal{D}^\circ}$ is finite over \mathcal{O} . There does not seem to be an obvious way to deduce the finiteness of $R_{\mathcal{D}^\circ}$ from that of $\tilde{R}_{\mathcal{D}^\circ}$; however, one can run the entirety of the above argument using the non-tilde rings to conclude that $R_{\mathcal{D}^\circ}$ is finite over \mathcal{O} . \square

4. MODULARITY LIFTING THEOREMS

(4.1) We now give our first modularity lifting theorem. By a “Hilbert eigenform with coefficients in $\overline{\mathbb{Q}_p}$ ” we mean a Hilbert eigenform together with an embedding of its coefficient field into $\overline{\mathbb{Q}_p}$. For such a form f , unramified outside a finite set of primes S , we write t_f for the type function on S of its associated Galois representation ρ_f (see §2.6). Note that t_f is definite by Proposition 2.6.1.

Theorem 4.1.1. *Let F be a totally real number field, S a finite set of primes and $\rho : G_{F,S} \rightarrow \text{GL}_2(\overline{\mathbb{Q}_p})$ an odd, weight two representation such that $\bar{\rho}$ satisfies (A1) and (A2) of §3. Assume there exists a parallel weight two Hilbert eigenform f with coefficients in $\overline{\mathbb{Q}_p}$ such that f is unramified outside S , $\bar{\rho} \cong \bar{\rho}_f$ and $t_\rho \approx t_f$. Then there is a Hilbert eigenform g with coefficients in $\overline{\mathbb{Q}_p}$ such that $\rho \cong \rho_g$.*

Proof. We call an extension F'/F *pre-solvable* if its Galois closure is solvable. This property behaves well in towers (unlike the property of being solvable, which requires Galois) and under compositum. By Langlands’ base change, it suffices to prove that ρ is modular after a pre-solvable extension. To begin with, we can pass to a totally real pre-solvable extension so that the following condition holds:

- $\det \rho$ and $\det \rho_f$ are of the form $\psi^2 \chi_p$ (with possibly different ψ).

(This is proved in Corollary 4.2.2 below.) We can thus replace ρ and ρ_f with twists so that they have determinant χ_p . Pick a finite extension E/\mathbb{Q}_p such that ρ and ρ_f are defined over E . After making another totally real pre-solvable base change, and possibly enlarging E , we can now ensure the following:

- $\bar{\rho}$ still satisfies (A1) and (A2) from §3.
- $\bar{\rho}$ is everywhere unramified.
- ρ and ρ_f are admissible at all finite places.
- $\bar{\rho}$ is trivial at all places where ρ ramifies and all places above p .
- The set of primes at which ρ_f have type C has even cardinality.
- F has even degree over \mathbb{Q} .
- k contains the eigenvalues of $\bar{\rho}$.

We thus see that $\bar{\rho}$ satisfies (A1)–(A7) of §3.1. Note that if $v \nmid p$ is a place at which ρ ramifies then the image of inertia at v under ρ and ρ_f is unipotent and $\pi_{f,v}$ is special of conductor one.

We now define a deformation datum $\mathcal{D}^\circ = (t, \Sigma^{\text{ram}}, \Sigma^{\text{aux}})$. We let Σ^{ram} be the set of primes away from p at which ρ_f ramifies. We let $t = t_f$. We let Σ^{aux} be the set $\{w\}$ where w is any large place of F not in $\Sigma_p \cup \Sigma^{\text{ram}}$ such that $\mathbf{N}w \not\equiv 1 \pmod{p}$ and $\text{tr } \bar{\rho}(\text{Frob}_w) \neq \pm(1 + \mathbf{N}w)$. The existence of such a w is

guaranteed by [DDT, Lemma 4.11] since $\bar{\rho}$ satisfies (A1) and the order of $\bar{\chi}_p$ is even. Conditions (A8) and (A9) of §3.3 are fulfilled.

Now let D be the unique quaternion algebra over F ramified at the infinite places and at Σ_C . Let U° be a standard compact open which is maximal everywhere except at w and satisfies $(*)$. Let \mathfrak{m} be the maximal ideal of $\mathbb{T}(U^\circ)$ determined by f (after applying the Jacquet-Langlands correspondence). Then U° and \mathfrak{m} satisfy conditions (B1)–(B6) of §3.7, after possibly enlarging k . (As remarked there, conditions (B4) and (B5) follow from the fact $t = t_f$.) We now apply Theorem 3.11.1 to conclude that $R_{\mathcal{D}^\circ}^\square \otimes_{\mathcal{O}} E = \mathbb{T}_{\mathcal{D}^\circ}^\square \otimes_{\mathcal{O}} E$. (Recall that $R_{\mathcal{D}^\circ}^\square[1/p] = \tilde{R}_{\mathcal{D}^\circ}^\square[1/p]$.) The representation ρ gives a $\overline{\mathbb{Q}}_p$ -point of $R_{\mathcal{D}^\circ}^\square$ (this is where we use $t_\rho \approx t_f$). The corresponding $\overline{\mathbb{Q}}_p$ -point of $\mathbb{T}_{\mathcal{D}^\circ}^\square$ defines the requisite Hilbert eigenform g . \square

(4.2) We now give a proof of the elementary fact concerning characters used in the first step of the proof of Theorem 4.1.1. We thank Bhargav Bhatt for the proof.

Lemma 4.2.1. *Let F be a number field and let $\psi : G_F \rightarrow \mathbb{Q}/\mathbb{Z}$ be a finite order character. Then ψ is the square of another character if and only if it is locally at all places.*

Proof. We have an exact sequence of abelian groups

$$0 \longrightarrow \mathbb{Z}/2\mathbb{Z} \longrightarrow \mathbb{Q}/\mathbb{Z} \xrightarrow{2} \mathbb{Q}/\mathbb{Z} \longrightarrow 0.$$

Regarding these as trivial Galois modules and taking cohomology gives

$$\begin{array}{ccccc} \mathrm{Hom}(G_{F_v}, \mathbb{Q}/\mathbb{Z}) & \xrightarrow{2} & \mathrm{Hom}(G_{F_v}, \mathbb{Q}/\mathbb{Z}) & \longrightarrow & \mathrm{Br}(F_v)[2] \\ \uparrow & & \uparrow & & \uparrow \\ \mathrm{Hom}(G_F, \mathbb{Q}/\mathbb{Z}) & \xrightarrow{2} & \mathrm{Hom}(G_F, \mathbb{Q}/\mathbb{Z}) & \longrightarrow & \mathrm{Br}(F)[2] \end{array}$$

where $\mathrm{Br}(\cdot)[2]$ is the 2-torsion of the Brauer group. Let ψ be a character of G_F and α the class in $\mathrm{Br}(F)[2]$ given by the above map. If ψ is locally a square then α maps to zero in each $\mathrm{Br}(F_v)[2]$. This implies that $\alpha = 0$ and so ψ is a square. \square

Corollary 4.2.2. *Let F be a totally real field, let M/F be a finite extension and let $\psi : G_F \rightarrow \mathbb{Q}/\mathbb{Z}$ be a finite order character such that $\psi(c) = 1$ for each complex conjugation c . Then there exists a finite, totally real, pre-solvable extension F'/F , linearly disjoint from M , such that $\psi|_{G_{F'}} = \eta^2$ where $\eta : G_{F'} \rightarrow \mathbb{Q}/\mathbb{Z}$ is an unramified character.*

Proof. Pick a finite, totally real, pre-solvable extension F_1/F linearly disjoint from M such that $\psi|_{G_{F_1}}$ is everywhere unramified. Then $\psi|_{G_{F_1}}$ is locally a square since at any finite place it can be regarded as a character of $\widehat{\mathbb{Z}}$ (and any map $\widehat{\mathbb{Z}} \rightarrow \mathbb{Q}/\mathbb{Z}$ is a square) while at infinite places it is trivial (and thus a square). Thus $\psi|_{G_{F_1}} = \eta^2$. Now pick a finite, totally real, pre-solvable extension F'/F_1 linearly disjoint from M such that $\eta|_{G_{F'}} is everywhere unramified. $\square$$

(4.3) The hypothesis $t_\rho \approx t_f$ occurring in Theorem 4.1.1 is more restrictive than one would like. We now examine ways of removing or relaxing it. We begin by stating the following conjecture:

Conjecture 4.3.1. *Let F be a totally real field, f a cuspidal parallel weight two Hilbert eigenform with coefficients in $\overline{\mathbb{Q}}_p$ unramified outside of some finite set S , t a definite type function on S and M/F a finite extension. Then there exists a finite, solvable, totally real extension F'/F , linearly disjoint from M , and a cuspidal parallel weight two Hilbert eigenform f' over F' with coefficients in $\overline{\mathbb{Q}}_p$ such that $t_{f'} = t|_{F'}$ and $\bar{\rho}_{f'} = \bar{\rho}_f|_{G_{F'}}$.*

The results of §7 can be used to show that the analogous conjecture holds on the Galois side. The condition $t_{f'} = t|_{F'}$ in the conjecture is equivalent to $t_{f'} \approx t|_{F'}$ as each type function is definite. If this conjecture held then we could remove the $t_\rho \approx t_f$ condition from Theorem 4.1.1. Indeed, say $\bar{\rho} = \bar{\rho}_f$ is given. By the conjecture, one can then make a solvable base change F'/F so that $\bar{\rho}|_{G_{F'}} = \bar{\rho}_{f'}$ and $t_\rho|_{F'} \approx t_{f'}$. Theorem 4.1.1 then gives that $\rho|_{G_{F'}}$ is modular. Finally, Langlands' base change gives that ρ is modular.

In the following sections, we will establish certain instances of Conjecture 4.3.1. Each such instance will give a strengthening of Theorem 4.1.1, as outlined in the previous paragraph.

(4.4) We now establish Conjecture 4.3.1 away from p . Precisely, we prove the following:

Proposition 4.4.1. *Conjecture 4.3.1 is true if $\bar{\rho}_f$ satisfies (A1) and $t_f|_{\Sigma_p} = t|_{\Sigma_p}$.*

In fact, [Ki, §3.5] essentially contains a proof of this result. We follow that section very closely, making only the necessary minor changes. Before getting into the proof, we note that the proposition gives Theorem 1.1.4, which we restate in our present language:

Theorem 4.4.2. *Let $\rho : G_F \rightarrow \mathrm{GL}_2(\overline{\mathbb{Q}}_p)$ an odd, finitely ramified, weight two representation such that $\bar{\rho}$ satisfies (A1) and (A2) of §3. Assume there exists a parallel weight two Hilbert eigenform f with coefficients in $\overline{\mathbb{Q}}_p$ such that $\bar{\rho} \cong \bar{\rho}_f$ and $t_\rho|_{\Sigma_p} \approx t_f|_{\Sigma_p}$. Then there is a Hilbert eigenform g with coefficients in $\overline{\mathbb{Q}}_p$ such that $\rho \cong \rho_g$.*

We now begin on the proof of Proposition 4.4.1. We need a few lemmas.

Lemma 4.4.3. *Let f be a parallel weight two Hilbert eigenform over F with coefficients in $\overline{\mathbb{Q}}_p$ and trivial nebentypus and let M/F be a finite extension. Assume that $\bar{\rho}_f$ satisfies (A1). Then there exists a finite, pre-solvable, totally real extension F'/F linearly disjoint from M and a parallel weight two Hilbert modular form f' over F' with coefficients in $\overline{\mathbb{Q}}_p$ and trivial nebentypus such that $\bar{\rho}_{f'} \cong \bar{\rho}_f|_{G_{F'}}$; $t_{f'}(v) = (t_f|_{F'})(v)$ at places v above p ; and $\pi_{f'}$ is special of conductor one at the primes above p for which $\rho_{f'}$ is type C , and unramified everywhere else.*

Proof. By replacing F with a finite, pre-solvable, totally real extension we may assume that $\bar{\rho}_f$ is everywhere unramified, F has even degree over \mathbb{Q} , the set of places above p at which ρ_f has type C has even cardinality and π_f is either unramified or special of conductor one at all places. We now explain the small modifications to [SW] needed to prove the lemma. Let D be the unique quaternion algebra over F ramified at the infinite places and at the places above p where f has type C . We use this division algebra instead of the one used in [SW]. We modify the definition of the Hecke algebra \mathbf{T} used in [SW] on p. 21 as in the proof of [Ki, Lemma 3.5.2] except that we only include the Hecke operators at places above p where ρ_f has type A or B . The proof now continues unchanged. \square

We now describe a level-raising result we will need. Let D be a quaternion algebra over F ramified at all the infinite places and some set of finite places. Let U be a standard maximal compact subgroup satisfying $(*)$. Let w be a finite place of F not above p , at which D is unramified and at which U_w is maximal compact. Let U' be the standard compact subgroup given by $U'_v = U_v$ for $v \neq w$ and

$$U'_w = \Gamma_0(w) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2(\mathcal{O}_{F_w}) \mid c \in \mathfrak{p}_w \right\}$$

Define a map

$$i : S_2(U)^{\oplus 2} \rightarrow S_2(U'), \quad (f_1, f_2) \mapsto f_1 + \begin{pmatrix} 1 & \\ & \varpi_w \end{pmatrix} f_2.$$

We now have the relevant result:

Lemma 4.4.4. *Notation being as above, let \mathfrak{m} be a non-Eisenstein maximal ideal of $\mathbf{T}(U)$. Assume that $T_w = \pm(\mathbf{N}w + 1) \pmod{\mathfrak{m}}$. Then $S_2(U')_{\mathfrak{m}}/\mathrm{im} i$ is a non-zero free \mathcal{O} -module.*

Proof. The proof goes the same as the one in [Ki, §3.1.7, §3.1.9]. \square

We prove one more lemma:

Lemma 4.4.5. *Let f be a parallel weight two Hilbert modular form over F with coefficients in $\overline{\mathbb{Q}}_p$ and trivial nebentypus. Assume that F has even degree over \mathbb{Q} , $\bar{\rho}_f$ satisfies (A1), the set of primes over p at which ρ_f is type C has even cardinality and π_f is unramified away from p . Let M/F be a finite extension, let $S = \{v_1, \dots, v_r\}$ be a set of primes of f disjoint from Σ_p at which $\bar{\rho}_f$ is trivial and such that $\mathbf{N}v_i = 1 \pmod{p}$ and let w be a place of F not in $\Sigma_p \cup S$. We can then find a tower of fields $F = F_0 \subset \dots \subset F_r$ and for each $i \in \{0, \dots, r\}$ a parallel weight two Hilbert modular form f_i over F_i with coefficients in $\overline{\mathbb{Q}}_p$ and trivial nebentypus such that:*

- (1) F_i/F_{i-1} is finite, abelian and totally real.
- (2) F_i is linearly disjoint from M .
- (3) w splits completely in F_i .
- (4) There are an even number of primes in F_i lying above $\{v_1, \dots, v_i\}$.

- (5) The primes $\{v_{i+1}, \dots, v_r\}$ are inert in F_i .
- (6) There are an even number of primes in F_i above p at which ρ_{f_i} has type C .
- (7) $\bar{\rho}_{f_i} \cong \bar{\rho}_f|_{G_{F_i}}$
- (8) $(t_{f_i})(v) = (t_f|_{F_i})(v)$ at places v above p .
- (9) π_{f_i} is special of conductor one at the primes over $\{v_1, \dots, v_i\}$ and the primes above p at which ρ_f is type C and unramified at all other primes except possibly those lying over w .

Proof. Replace M with the compositum of M and $\ker \bar{\rho}_f$. We follow [Ki, Lemma 3.5.3] and prove the lemma by induction. Thus assume we have constructed $F_0 \subset \dots \subset F_{i-1}$ and f_0, \dots, f_{i-1} . Let D be the unique quaternion algebra over F_{i-1} ramified at the infinite places, the places above p at which $\rho_{f_{i-1}}$ has type C and the primes lying above $\{v_1, \dots, v_{i-1}\}$. Let U be a standard compact subgroup satisfying the following: U is maximal everywhere except for the primes above w ; U satisfies $(*)$; f_{i-1} corresponds to an eigenform in $S_2(U)$ under the Jacquet-Langlands correspondence (which we will still denote by f_{i-1}). Let U' be the standard compact given by $U'_v = U_v$ unless v is the unique prime of F_{i-1} lying over v_i , in which case $U'_v = \Gamma_0(v)$. Let \mathfrak{m} be the maximal ideal of $\mathbb{T}(U)$ determined by f . Since $\bar{\rho}_f$ is trivial at v_i we have

$$\mathrm{tr} \bar{\rho}_f(\mathrm{Frob}_{v_i}) = T_{v_i} = 2 \pmod{\mathfrak{m}}, \quad \text{and} \quad \det \bar{\rho}_f(\mathrm{Frob}_{v_i}) = \mathbf{N} v_i = 1 \pmod{\mathfrak{m}}.$$

As $\mathbf{N} v_i = 1$ we see that the condition of Lemma 4.4.4 is satisfied and so $S_2(U')_{\mathfrak{m}}/\mathrm{im} i$ is a non-zero free \mathcal{O} -module. We can therefore pick a non-zero eigenform f'_i in $S_2(U')$ not in the image of i and an embedding of the coefficient field of f'_i into $\overline{\mathbb{Q}}_p$ such that $\bar{\rho}_{f'_i} \cong \bar{\rho}_{f_{i-1}}$. Because f'_i is a newform for U' it follows that $\pi_{f'_i}$ will be special at all the primes in $\{v_1, \dots, v_i\}$ and at the primes above p at which $\rho_{f'_i}$ have type C and unramified everywhere else except possibly for the primes over w . In particular, $\rho_{f'_i}$ has type C wherever $\rho_{f_{i-1}}$ does. Looking at the Hecke operators at primes above p at which $\rho_{f_{i-1}}$ does not have type C shows that $\rho_{f'_i}$ and $\rho_{f_{i-1}}$ have the same type at these primes.

Now pick a finite, totally real, abelian extension F_i/F_{i-1} satisfying (1)-(6). Let f_i be the base change of f'_i to F_i . Then F_i and f_i satisfy (1)-(9). We have therefore established the lemma. \square

We now return to the proof of the proposition.

Proof of Proposition 4.4.1. Replace M with the compositum of M and $\ker \bar{\rho}_f$. To begin with, we may use Corollary 4.2.2 to find a finite, totally real, pre-solvable extension F_1/F linearly disjoint from M so that $\det \rho_f|_{G_{F_1}} = \psi^2 \chi_p$. Let $f_1 = \psi^{-1} \cdot f|_{F_1}$; this form has trivial nebentypus. We can now apply Lemma 4.4.3 to produce a finite, totally real, pre-solvable extension F_2/F_1 linearly disjoint from M and a form f_2 over F_2 so that: $\bar{\rho}_{f_2} \cong \bar{\rho}_{f_1}|_{G_{F_2}}$; $t_{f_1}(v) \approx t_{f_2}(v)$ at all places v above p ; F_2 has even degree over \mathbb{Q} ; the set of places of F_2 above p at which f_2 has type C has even cardinality; $\bar{\rho}_{f_2}$ is everywhere unramified; $\bar{\rho}_{f_2}$ is trivial at all places in S ; we have $\mathbf{N} v = 1 \pmod{p}$ for all v above $S \setminus \Sigma_p$; and π_{f_2} is unramified except at the places above p where ρ_{f_2} has type C , where it is special of conductor one. Choose a place w of F_2 above S such that

$$\mathrm{tr} \bar{\rho}(\mathrm{Frob}_w) \neq \pm(1 + \mathbf{N} w),$$

This is possible by [DDT, Lemma 4.11], as before. We now apply Lemma 4.4.5 to produce a finite, totally real, pre-solvable extension F_3/F_2 linearly disjoint from M and a form f_3 over F_3 such that: $\bar{\rho}_{f_3} = \bar{\rho}_{f_2}|_{G_{F_3}}$; $t_{f_3} \approx t|_{F_3}$ at all places v above p ; w splits in F_3 . Since w satisfies the above condition and is split in F_3 it follows that f is not special at any place above w . Therefore, we can find a finite, totally real, pre-solvable extension F_4/F_3 linearly disjoint from M such that if $f_4 = f_3|_{F_4}$ then we can make π_{f_4} unramified at all places above w and still satisfies the other properties we want. We now take $F' = F_4$ and $f' = \psi \cdot f_4$. \square

(4.5) We now discuss some instances of Conjecture 4.3.1 at places above p . These results will not be used in the remainder of the paper. To begin with, we have the following:

Proposition 4.5.1. *Conjecture 4.3.1 holds if t_f only assumes the values A and B above p and t only assumes the value B above p .*

Proof. See [Ki, Theorem 3.5.7]. \square

We now show, using some results of §7 and a modularity lifting theorem of Skinner-Wiles, that in certain circumstances one can switch between types A and C . We first need some terminology. Let F_v/\mathbb{Q}_p be a finite extension, let $\bar{\rho}_v : G_{F_v} \rightarrow \mathrm{GL}_2(\overline{\mathbb{F}}_p)$ be a representation and let $*$ be A or C . We say that $(\bar{\rho}_v, *)$ is *good*

if there exists a finite extension F'_v/F_v and a lift ρ_v of $\bar{\rho}_v|_{G_{F'_v}}$ such that ρ_v is weight two and has definite type $*$ and we have

$$\bar{\rho}_v|_{G_{F'_v}} = \begin{pmatrix} \bar{\alpha} & * \\ & \bar{\beta} \end{pmatrix}, \quad \rho_v = \begin{pmatrix} \alpha & * \\ & \beta \end{pmatrix}$$

with α reducing to $\bar{\alpha}$ and β to $\bar{\beta}$ and where $\bar{\alpha} \neq \bar{\beta}$.

Proposition 4.5.2. *Conjecture 4.3.1 holds if $\bar{\rho}_f$ satisfies (A1), t_f and t only assume the values A and C at primes above p and for each place $v \mid p$ the pair $(\bar{\rho}_f|_{G_{F'_v}}, t(v))$ is good.*

Proof. For $v \mid p$ let F'_v/F_v and ρ_v be the extension and lift provided by the goodness hypothesis. Let τ_v be the inertial type of ρ_v (see §7). Pick a finite, pre-solvable, totally real extension F'/F such that $(F')_v = F'_v$ for $v \mid p$. Define a lifting problem (see §7) $\mathcal{P} = (\Sigma, \psi, t|_{F'}, \{\tau_v\})$ over F' as follows. The character ψ is just the nebentypus of f restricted to $G_{F'}$. The set Σ consists of the primes at which $\bar{\rho}_f|_{G_{F'}}$ and ψ ramify together with all the primes over p . The type function is specified. Finally, the inertial types τ_v have been specified for v above p ; for the other places, just take τ_v to be the inertial type of $\rho_f|_{G_{F'}}$ at v . By definition (and the fact that t_f is definite), \mathcal{P} is locally solvable. By Theorem 7.2.1 it is therefore solvable. Let ρ' be a solution. As $\bar{\rho}' = \bar{\rho}_f|_{G_{F'}}$ the representation ρ' is residually modular. Furthermore, the conditions we have imposed ensures that it satisfies the conditions of [SW2, Theorem 5.1]. We therefore conclude that ρ' is modular, i.e., of the form $\rho_{f'}$. The extension F'/F and the form f' give the required output of Conjecture 4.3.1. \square

Remark 4.5.3. The above proposition allows one to perform level raising or level lowering at p in many cases. For instance, one can use it to find mod p congruences between forms which are special at p and forms which are not.

5. POTENTIAL MODULARITY

(5.1) In §5 we prove the following two potential modularity theorems. The second of these theorems is a stronger and more precise version of Theorem 1.1.1. These are simple modifications of Taylor's original result [Tay3]. In fact, our proof of potentially modularity is simpler than Taylor's since we have a stronger modularity lifting theorem at our disposal.

Theorem 5.1.1. *Let $\bar{\rho} : G_F \rightarrow \mathrm{GL}_2(\bar{\mathbb{F}}_p)$ be any odd representation. Let $\psi : G_F \rightarrow \mathcal{O}^\times$ be a finite order character such that $\det \bar{\rho} = \bar{\psi} \cdot \bar{\chi}_p$, let M/F be a finite extension and let t be a definite type function on Σ_p . Then there exists a finite Galois extension M'/F containing M and a finite, totally real Galois extension F'/F linearly disjoint from M' such that for any finite, totally real extension F''/F' linearly disjoint from M' there exists a cuspidal parallel weight two Hilbert eigenform f over F'' with coefficients in $\bar{\mathbb{Q}}_p$ such that $\det \rho_f = \psi \chi_p|_{G_{F''}}$, $\bar{\rho}_f \cong \bar{\rho}|_{G_{F''}}$ and $t_f|_{\Sigma_p} = t|_{F''}$.*

Theorem 5.1.2. *Let $\rho : G_F \rightarrow \mathrm{GL}_2(\bar{\mathbb{Q}}_p)$ be a finitely ramified, odd, weight two representation such that $\bar{\rho}$ satisfies (A1) and (A2) of §3. Let M/F be a finite extension. Then there exists a finite Galois extension M'/F containing M and a finite, totally real, Galois extension F'/F linearly disjoint from M' such that for any finite, totally real extension F''/F' linearly disjoint from M' there exists a cuspidal parallel weight two Hilbert eigenform f over F'' with coefficients in $\bar{\mathbb{Q}}_p$ such that $\rho_f \cong \rho|_{G_{F''}}$.*

In §8 we will improve these results and show that F' can be taken to be split at a given finite set of primes.

(5.2) Following [Tay3], we will prove Theorem 5.1.1 by using a theorem of Moret-Bailly. We now recall that theorem. Let F be any number field. A *Skolem datum* over F is a tuple $(X, \Sigma, \{L_v\}_{v \in \Sigma}, \{\Omega_v\}_{v \in \Sigma})$ consisting of:

- A geometrically connected, separated, smooth scheme X of finite type over F .
- A finite set of places Σ of F .
- For each $v \in \Sigma$ a Galois extension L_v of F_v .
- For each $v \in \Sigma$ a non-empty, Galois stable, open subset Ω_v of $X(L_v)$.

Here we regard $X(L_v)$ as having the v -adic topology. Our definition of a Skolem datum is less general than the one given in [MB].

Proposition 5.2.1 ([MB]). *Let a Skolem datum as above be given. Then there exists a finite Galois extension F'/F which splits over each L_v (that is, $F' \otimes_F L_v$ is a direct sum of L_v 's) and a point $x \in X(F')$ such that for each $v \in \Sigma$ and each embedding $F' \rightarrow L_v$ the image of x in $X(L_v)$ belongs to Ω_v .*

It will be convenient to give a slightly improved version of the theorem here. We learned of this improved version, and the proof, from [HSBT, Proposition 2.1] (they state that L. Dieulefait had made this observation as well).

Proposition 5.2.2. *Let a Skolem datum be given and let M/F be a finite extension. Then one can take the field F' supplied by Proposition 5.2.1 to be linearly disjoint from M .*

Proof. As X is smooth and geometrically connected, it has F_v -points for v large (the Weil conjectures give mod v points and then smoothness gives F_v points). Choose a finite set of places Σ' satisfying the following:

- The set Σ' is disjoint from Σ .
- The extension M/F is unramified everywhere above Σ' .
- For $v \in \Sigma'$ the set $X(F_v)$ is non-empty.
- The set $\{\text{Frob}_v\}_{v \in \Sigma'}$ generates $\text{Gal}(M'/F)$, where M' is the Galois closure of M .

For $v \in \Sigma'$ let $L_v = F_v$ and $\Omega_v = X(F_v)$, which is non-empty by construction. Then $(X, \Sigma \cup \Sigma', \{L_v\}, \{\Omega_v\})$ is a Skolem datum. By Proposition 5.2.1 we get an extension F'/F which splits over each L_v and a point $x \in X(F')$ such that for each embedding $F' \rightarrow L_v$ the image of x lands in Ω_v . As F' splits over $L_v = F_v$ for $v \in \Sigma'$ it follows that F' is linearly disjoint from M . \square

(5.3) The following proposition is our main application of the theorem of Moret-Bailly.

Proposition 5.3.1. *Let F and K be totally real number fields. Choose the following:*

- A finite extension M/F .
- Primes \mathfrak{p} and \mathfrak{l} of K over distinct odd rational primes p and ℓ .
- Representations $\bar{\rho}_{\mathfrak{p}} : G_F \rightarrow \text{GL}_2(k_{\mathfrak{p}})$ and $\bar{\rho}_{\mathfrak{l}} : G_F \rightarrow \text{GL}_2(k_{\mathfrak{l}})$ with cyclotomic determinant.
- Type functions $t_{\mathfrak{p}} : \Sigma_p \rightarrow \{A, B, C\}$ and $t_{\mathfrak{l}} : \Sigma_{\ell} \rightarrow \{A, B, C\}$.

We can then find a finite, totally real, Galois extension F'/F linearly disjoint from M and a $\text{GL}_2(K)$ -type abelian variety A/F' such that $A[\mathfrak{p}] \cong \bar{\rho}_{\mathfrak{p}}|_{G_{F'}}$ and $T_{\mathfrak{p}}A$ has type $t_{\mathfrak{p}}|_{F'}$, and similarly with \mathfrak{l} in place of \mathfrak{p} .

Proof. Pick a prime \mathfrak{r} of K above a prime $r \neq p, \ell$ such that the kernel of the reduction map $\text{GL}_2(\mathcal{O}_K) \rightarrow \text{GL}_2(k_{\mathfrak{r}})$ is torsion-free. Pick a representation $\bar{\rho}_{\mathfrak{r}} : G_F \rightarrow \text{GL}_2(k_{\mathfrak{r}})$ with cyclotomic determinant, e.g., $\bar{\rho}_{\mathfrak{r}} = \bar{\chi}_r \oplus 1$. Let V_w be the vector space schemes over F corresponding to $\bar{\rho}_w$ for $w \in \{\mathfrak{p}, \mathfrak{l}, \mathfrak{r}\}$. For each w , fix an isomorphism $a_w : \bigwedge^2 V_w \rightarrow k_w(1)$; this is the same as fixing an isomorphism of V_w with its Cartier dual. Let X be the scheme over F classifying tuples $(A, i, \{\alpha_w\}_w)$ where:

- A is a $\text{GL}_2(K)$ -type abelian variety, that is, an abelian variety of dimension $2[K : \mathbb{Q}]$ together with an embedding of \mathcal{O}_K into $\text{End}(A)$.
- $i : \mathcal{P}(A) \rightarrow \mathcal{D}^{-1}$ is an isomorphism of \mathcal{O}_K -modules equipped with a notion of positivity. Here $\mathcal{P}(A)$ is the set of isogenies $A \rightarrow A^{\vee}$, its set of positive elements being the polarizations; \mathcal{D}^{-1} is the inverse different of K , its set of positive elements being the totally real elements it contains.
- $\alpha_w : A[w] \rightarrow V_w$ is an isomorphism of \mathcal{O} -modules under which the \mathcal{O} -linear Weil pairing on the source coincides with the fixed isomorphisms a_w , for each $w \in \{\mathfrak{p}, \mathfrak{l}, \mathfrak{r}\}$.

Note that X exists as a scheme (rather than as a stack) because the kernel of $\text{GL}_2(\mathcal{O}_K) \rightarrow \text{GL}_2(\mathcal{O}_K/\mathfrak{p}\mathfrak{l}\mathfrak{r})$ is torsion-free. For more details on the definition (which will not be important to us), and the proof that X exists, see [Rap, §1], [Tay3, §2] or [Tay, §4].

We now sketch an argument to show that X is smooth and geometrically connected. To begin with, there is a similar moduli problem one can consider, which we will denote by X' , consisting of tuples $(A, i, \{\alpha_w\}_w)$ where A and i are as above but α_w is an \mathcal{O} -linear isomorphism $A[w] \rightarrow k_w^2$. We have not given all the details of the definition of the space X' , but it is meant to be the same as the space $\mathcal{M}_n^{\mathcal{D}^{-1}}$ of [Rap, §1], with $n = \mathfrak{p}\mathfrak{l}\mathfrak{r}$. (Note that Rapoport only considers the case where n is an integer, but the arguments apply to the case of ideals of \mathcal{O}_K as well). The spaces X and X' are nearly twisted forms of each other, but not quite. To elaborate on this, fix an embedding $F \rightarrow \mathbb{C}$ and trivializations of each $(V_w)_{\mathbb{C}}$. We then have a natural map $X_{\mathbb{C}} \rightarrow X'_{\mathbb{C}}$. This map is not surjective as in X' the only condition on α_w is that it be \mathcal{O} -linear while in X it must be \mathcal{O} -linear and also take the Weil pairing to the given pairing a_w . This is the only failure of

surjectivity, however, and one sees that $X_{\mathbb{C}} \rightarrow X'_{\mathbb{C}}$ identifies the former with the locus in the latter where the α_w take the Weil pairing to the a_w . Using the computation of the connected components of $X'_{\mathbb{C}}$ given in [Rap, Theorem 1.28 (ii)], one thus sees that $X_{\mathbb{C}}$ is a component of $X'_{\mathbb{C}}$. It is therefore smooth since $X'_{\mathbb{C}}$ is (see the discussion of [Rap, §1], in particular [Rap, Theorem 1.20]). We have thus shown that X/F is smooth and geometrically connected.

Let $\Sigma = \Sigma_p \cup \Sigma_\ell \cup \Sigma_\infty$. We now define a finite Galois extension L_v/F_v and a Galois stable open subset Ω_v of $X(L_v)$ for each $v \in \Sigma$, as follows:

- Say $v \in \Sigma_p$. If $t_p(v)$ equals A (resp. B) let $\overline{\Omega}_v$ be the subset of $X(\overline{F}_v)$ consisting of those abelian varieties over \overline{F}_v which have good reduction and whose reduction is ordinary (resp. non-ordinary) at p . If $t_p(v)$ equals C then take $\overline{\Omega}_v$ to be the subset of $X(\overline{F}_v)$ consisting of those abelian varieties which have bad reduction. Let L_v/F_v be any finite Galois extension for which $\overline{\Omega}_v \cap X(L_v)$ is non-empty and take Ω_v to be this intersection.
- For $v \in \Sigma_\ell$ we define L_v and Ω_v in exactly the same manner.
- For $v \in \Sigma_\infty$ we take $L_v = F_v$ and $\Omega_v = X(L_v)$. One easily sees that Ω_v is non-empty.

We now have a Skolem datum $(X, \Sigma, \{L_v\}, \{\Omega_v\})$. Proposition 5.2.2 thus gives a finite Galois extension F'/F linearly disjoint from M and a point $x \in X(F')$ corresponding to a $\mathrm{GL}_2(K)$ -type abelian variety A/F' satisfying the conclusions of the theorem. \square

(5.4) We need one more result before proving Theorem 5.1.1.

Proposition 5.4.1. *Let F be a totally real field and ℓ an odd prime. Then there exists a finitely ramified, odd, weight two representation $\rho : G_F \rightarrow \mathrm{GL}_2(\mathbb{Q}_\ell)$ and a finite Galois extension M/F such that for any finite totally real extension F'/F linearly disjoint from M the representation $\overline{\rho}|_{G_{F'}}$ satisfies (A1) and (A2) (with p changed to ℓ) and $\rho|_{G_{F'}}$ comes from a cuspidal parallel weight two Hilbert eigenform.*

Before proving the proposition we require a lemma.

Lemma 5.4.2. *Let H/\mathbb{Q} be an imaginary quadratic extension in which ℓ splits. Let \mathfrak{l}^+ and \mathfrak{l}^- be the two primes above ℓ . We can then find a finitely ramified character $\psi_0 : G_H \rightarrow \overline{\mathbb{Q}}_\ell^\times$ such that $\psi_0|_{I_{H, \mathfrak{l}^+}} = \chi_\ell$ and $\psi_0|_{I_{H, \mathfrak{l}^-}} = 1$.*

Proof. Let \mathfrak{r} be a prime of H over a prime $r \neq \ell$ and put $S = \{\mathfrak{l}^+, \mathfrak{l}^-, \mathfrak{r}\}$. Let $U_{\mathfrak{r}}$ be a compact open subgroup of $\mathcal{O}_{H, \mathfrak{r}}^\times$ such that $U_{\mathfrak{r}} \cap \mathcal{O}_H^\times = 1$. The map

$$\mathcal{O}_{H, \mathfrak{l}^+}^\times \times \mathcal{O}_{H, \mathfrak{l}^-}^\times \rightarrow \mathbb{A}_H^\times / (H^\times H_\infty^\times U_{\mathfrak{r}} \prod_{v \notin S} \mathcal{O}_{H, v}^\times).$$

is injective and has open image. It follows that any character of the left group valued in $\overline{\mathbb{Q}}_\ell^\times$ extends to a character of the right group, since $\overline{\mathbb{Q}}_\ell^\times$ is injective. The result now follows from class field theory. \square

We now return to the proof of the proposition.

Proof of Proposition 5.4.1. Let H/\mathbb{Q} be an imaginary quadratic field in which ℓ splits completely. Let \mathfrak{l}^+ and \mathfrak{l}^- be the two places of H above ℓ . For each place v of F above ℓ label the two places of FH above v corresponding to \mathfrak{l}^+ and \mathfrak{l}^- as v^+ and v^- . Let $\psi_0 : G_H \rightarrow \overline{\mathbb{Q}}_\ell^\times$ be the character produced by the previous lemma. We can multiply $\psi_0|_{G_{FH}}$ by a finite order character of G_{FH} to obtain a character $\psi : G_{FH} \rightarrow \overline{\mathbb{Q}}_\ell^\times$ satisfying the following two conditions:

- For each place v of F over ℓ we have $\psi|_{I_{FH, v^+}} = \chi_\ell$ and $\psi|_{I_{FH, v^-}} = 1$.
- Let ψ' be the $\mathrm{Gal}(FH/F)$ conjugate of ψ . Then $\overline{\psi}|_{G_{FH(\zeta_\ell)}} \neq \overline{\psi'}|_{G_{FH(\zeta_\ell)}}$.

We put $\rho = \mathrm{Ind}_{FH}^F(\psi)$ and let M be the Galois closure over F of the field determined by $\ker(\overline{\rho}|_{F(\zeta_\ell)})$.

We now show that ρ has the requisite properties. It is clear that ρ is finitely ramified. We can think of ρ as given by $\overline{\mathbb{Q}}_\ell[G_F] \otimes_{\overline{\mathbb{Q}}_\ell[G_{FH}]} \overline{\mathbb{Q}}_\ell(\psi)$. As such, if g is any element of G_F which does not belong to G_{FH} then $e_1 = 1 \otimes 1$ and $e_2 = g \otimes 1$ form a basis for ρ . In this basis, we have

$$(1) \quad \rho(h) = \begin{pmatrix} \psi(h) & \\ & \psi'(h) \end{pmatrix}, \quad \rho(g) = \begin{pmatrix} & \psi(g^2) \\ 1 & \end{pmatrix}$$

where $h \in G_{FH}$. As any complex conjugation in G_F can be taken to be g , and these elements square to the identity, (1) shows that ρ is odd. We now check that ρ has weight two. It suffices to check this after a finite extension, so we may as well check that $\rho|_{G_{FH}}$ has weight two. If h is an element of I_{FH,v^+} then $g^{-1}hg$ belongs to I_{H,v^-} and conversely. Thus (1) shows that $\rho|_{I_{FH,v^+}}$ and $\rho|_{I_{FH,v^-}}$ are isomorphic to $\chi_\ell \oplus 1$, and so ρ has weight two.

We now prove the statements about $\bar{\rho}$. Choose an element g of $G_{F(\zeta_\ell)} \subset G_F$ which does not belong to G_{FH} , which is possible since H is disjoint from $F(\zeta_\ell)$, and let e_1, e_2 be a basis for ρ as above. The matrices in (1) belong to $\text{GL}_2(\overline{\mathbb{Z}}_\ell)$ and thus give an integral model for ρ . We define $\bar{\rho}$ to be the reduction of this integral model. From the first matrix in (1) and our second condition on ψ we see that $\bar{\rho}(G_{FH(\zeta_\ell)})$ is contained in the group of diagonal matrices but not in the group of scalar matrices. On the other hand, we have $\bar{\rho}(g) = \begin{pmatrix} & * \\ 1 & \end{pmatrix}$. These two conditions show that $\bar{\rho}(G_{F(\zeta_\ell)})$ is an irreducible subgroup of $\text{GL}_2(\overline{\mathbb{F}}_\ell)$.

Thus $\bar{\rho}$ satisfies (A1). The image of $\bar{\rho}$ is solvable and so (A2) is satisfied.

Now, since ρ is the induction of a character it comes from an automorphic form. As ρ is odd, weight two and absolutely irreducible, this automorphic form is a cuspidal, parallel weight two Hilbert modular form. Thus ρ is modular.

Finally, if F'/F is any extension linearly disjoint from M then $\rho|_{G_{F'}} = \text{Ind}_{F'H}^{F'}(\psi)$. As $\psi|_{G_{HF'}}$ has the same two properties that we imposed on ψ , the above arguments show that $\rho|_{G_{F'}}$ is modular and that $\bar{\rho}|_{G_{F'}}$ satisfies (A1) and (A2). This completes the proof. \square

(5.5) We can now complete proof of Theorem 5.1.1. Let $\bar{\rho}, \psi, M_1 = M$ and t be given as in the statement of the theorem. Pick an odd prime $\ell \neq p$ and let ρ' and M_2 be the representation and extension given by Proposition 5.4.1. Let M' be the Galois closure over F of the compositum of M_1 and M_2 . Write $\det \rho' = \chi_\ell \psi'$. Pick a totally real field K and primes \mathfrak{p} and \mathfrak{l} above p and ℓ such that the images of $\bar{\rho}$ and $\bar{\rho}'$ are contained in $\text{GL}_2(k_{\mathfrak{p}})$ and $\text{GL}_2(k_{\mathfrak{l}})$. Apply Corollary 4.2.2 to find a finite, totally real extension F_1/F , linearly disjoint from M' , such that ψ and ψ' are squares when restricted to G_{F_1} ; say $\psi|_{G_{F_1}} = \psi_1^2$ and $\psi'|_{G_{F_1}} = (\psi'_1)^2$. Put $\bar{\rho}_{\mathfrak{p}} = \psi_1^{-1} \cdot (\bar{\rho}|_{G_{F_1}})$ and $\bar{\rho}_{\mathfrak{l}} = (\psi'_1)^{-1} \cdot (\bar{\rho}'|_{G_{F_1}})$. Let $t_{\mathfrak{p}} = t$ and let $t_{\mathfrak{l}} = t_{\rho'}$.

We now apply Proposition 5.3.1 (with the M there being our M'). Let F_2/F_1 and A/F_2 be the resulting field and abelian variety. Let F''/F_2 be any finite, totally real, extension linearly disjoint from M' . We have $A[\mathfrak{l}]|_{G_{F''}} = (\psi'_1)^{-1} \cdot \bar{\rho}'|_{G_{F''}}$, which satisfies (A1) and (A2). Furthermore, $(T_{\mathfrak{l}}A)|_{G_{F''}}$ and $(\psi'_1)^{-1} \cdot \rho'|_{G_{F''}}$ are each finitely ramified, odd, weight two lifts of $A[\mathfrak{l}]|_{G_{F''}}$ which have the same type at each place of F'' over ℓ and $(\psi'_1)^{-1} \cdot \rho'|_{G_{F''}}$ is modular. We conclude from Theorem 4.4.2 that $(T_{\mathfrak{l}}A)|_{G_{F''}}$ is modular, that is, of the form $\rho_{f',\mathfrak{l}'}$ for some parallel weight two Hilbert eigenform f' over F'' and some prime \mathfrak{l}' of its coefficient field. Since $(T_{\mathfrak{l}}A)|_{G_{F''}}$ is absolutely irreducible, f must be cuspidal. By compatibility, we see that $(T_{\mathfrak{p}}A)|_{G_{F''}}$ is isomorphic to $\rho_{f',\mathfrak{p}'}$. Letting $f = \psi_1 \cdot f'$, we find $\bar{\rho}|_{G_{F''}} = \bar{\rho}_{f,\mathfrak{p}'}$ and $\det \rho_{f,\mathfrak{p}'} = \chi_p \psi|_{G_{F''}}$. Finally, since $\rho_{f,\mathfrak{p}'}$ is off from $(T_{\mathfrak{p}}A)|_{G_{F''}}$ by a finite order character, the two have the same type at all places over p . Since the type of the Tate module above p is prescribed by t by the way we chose A , so is the type of $\rho_{f,\mathfrak{p}'}$. Taking F' to be the Galois closure of F_2 over F finishes the proof.

(5.6) We now prove Theorem 5.1.2. Let ρ and $M = M_1$ be given as in the statement of the theorem. Let t be a definite type function on Σ_p with $t \approx t_{\rho}|_{\Sigma_p}$. Apply Theorem 5.1.1 to $\bar{\rho}, M$ and t . Let F' and M_2 be the resulting extensions. Let M' be the Galois closure over F of the compositum of M_1, M_2 and the field determined by $\ker \bar{\rho}$. Let F''/F' be a finite, totally real extension linearly disjoint from M' . Let f be a parallel weight two Hilbert modular form over F'' with coefficients in $\overline{\mathbb{Q}}_p$ such that $\bar{\rho}|_{G_{F''}} = \bar{\rho}_f$ and the type of ρ_f at places above p is given by t . The representation $\bar{\rho}|_{G_{F''}}$ still satisfies (A1) and (A2). We can now apply Theorem 4.4.2 to conclude that $\rho|_{G_{F''}}$ is modular. This completes the proof.

(5.7) We have now proved Theorem 1.1.1. We will prove Corollary 1.1.2 in §9. We remark that Proposition 2.6.1 now applies and shows that t_{ρ} is definite for any odd, finitely ramified, weight two ρ satisfying (A1) and (A2).

6. FINITENESS OF DEFORMATION RINGS

(6.1) Let F be a totally real field. Fix an odd representation $\bar{\rho} : G_F \rightarrow \text{GL}_2(k)$ satisfying (A1) and (A2). Also fix a finite order character $\psi : G_F \rightarrow \mathcal{O}^\times$ such that $\det \bar{\rho} = \bar{\psi} \cdot \overline{\chi}_p$. Let Σ be a finite set of primes of F , including all those above p and those at which $\bar{\rho}$ or ψ ramify. For $v \in \Sigma$ let R_v^\square denote the universal ring

classifying framed deformations of $\bar{\rho}|_{G_{F_v}}$ with determinant $\psi\chi_p$. Choose a definite type function t defined on Σ and for each $v \in \Sigma$ a quotient $R_v^{\square, \dagger}$ of R_v^{\square} . We require the following:

- (1) $R_v^{\square, \dagger}$ is non-zero, reduced and \mathcal{O} -flat of dimension $[F_v : \mathbb{Q}_p] + 4$ (resp. 4) for $v \in \Sigma_p$ (resp. $v \notin \Sigma_p$).
- (2) For each $v \in \Sigma$ there exists a finite Galois extension L_v/F_v such that given an extension E'/E and a point $R_v^{\square, \dagger} \rightarrow E'$ corresponding to a representation ρ we have that $\rho|_{L_v}$ is admissible of type $t(v)$.

As usual, we will write Σ_* for the inverse image of $*$ under t .

We now define the following deformation rings:

- We let $R^{\square, \text{loc}}$ be the tensor product of the R_v^{\square} for $v \in \Sigma$.
- We let $R^{\square, \text{loc}, \dagger}$ be the tensor product of the $R_v^{\square, \dagger}$ for $v \in \Sigma$.
- We let R_{Σ}^{\square} be the universal ring of $\bar{\rho}$ classifying deformations of $\bar{\rho}$ with determinant $\psi\chi_p$ which are unramified away from Σ together with a framing at the places in Σ .
- We put $R^{\square, \dagger} = R^{\square, \text{loc}, \dagger} \otimes_{R^{\square, \text{loc}}} R_{\Sigma}^{\square}$.
- We let R_{Σ} be the universal ring classifying deformations of $\bar{\rho}$ with determinant $\psi\chi_p$ unramified away from Σ .
- We let R^{\dagger} be the quotient of R_{Σ} analogous to $R^{\square, \dagger}$.

The main result of §6 is the following:

Theorem 6.1.1. *The ring R^{\dagger} is finite over \mathcal{O} of non-zero rank. In particular, the set $\text{Hom}(R^{\dagger}, \overline{\mathbb{Q}}_p)$ is finite and non-empty.*

(6.2) We begin the proof of Theorem 6.1.1 with the following.

Proposition 6.2.1. *The ring R^{\dagger} has dimension at least one.*

Proof. Since $\bar{\rho}$ satisfies condition (A1) we have $H^0(G_{F,S}, (\text{ad}^{\circ} \bar{\rho})^*(1)) = 0$. We can therefore apply [Ki3, Proposition 4.1.5] to conclude that there is an isomorphism

$$R_{\Sigma}^{\square} = R^{\square, \text{loc}}[[x_1, \dots, x_{r+n-1}]]/(f_1, \dots, f_{r+s})$$

where $s = \sum_{v \in \Sigma} \dim_k H^0(G_{F_v}, \text{ad}^{\circ} \bar{\rho})$, n is the cardinality of Σ and r is some non-negative integer. By tensoring this isomorphism over $R^{\square, \text{loc}}$ with $R^{\square, \text{loc}, \dagger}$, we find

$$R^{\square, \dagger} = R^{\square, \text{loc}, \dagger}[[x_1, \dots, x_{r+n-1}]]/(f_1, \dots, f_{r+s})$$

Now, the assumption that $\bar{\rho}$ is odd gives $s = [F : \mathbb{Q}]$. The ring $R^{\square, \text{loc}, \dagger}$ has dimension $d + 1$ where

$$d = \sum_{v \in \Sigma_p} ([F_v : \mathbb{Q}_p] + 3) + \sum_{v \in \Sigma \setminus \Sigma_p} 3 = [F : \mathbb{Q}] + 3n$$

We thus find

$$\dim R^{\square, \dagger} \geq ([F : \mathbb{Q}] + 3n + 1) + (r + n - 1) - (r + s) = 4n$$

As $R^{\dagger} \rightarrow R^{\square, \dagger}$ is formally smooth of relative dimension $4n - 1$ we conclude $\dim R^{\dagger} \geq 1$. \square

(6.3) In light of Proposition 6.2.1, to prove Theorem 6.1.1 it suffices to show that R^{\dagger} is finite over \mathcal{O} . To do this we will need to use the following lemma:

Lemma 6.3.1. *Let F'/F be a finite extension for which $\bar{\rho}|_{G_{F'}}$ is absolutely irreducible. Let \bar{R}_{Σ} be the universal ring classifying deformations of $\bar{\rho}|_{G_{F'}}$ unramified outside of the primes above Σ . Then the map $\bar{R}_{\Sigma} \rightarrow R_{\Sigma}$ is finite.*

Proof. We sketch a proof. As the rings involved are topologically finitely generated, it suffices to show that the map is integral. Deformation rings are generated by the traces of elements of the group under the universal representation, so the result thus follows from the following statement: if $\rho : G \rightarrow \text{GL}_2(R)$ is a representation and g belongs to G then $\text{tr } \rho(g)$ satisfies a monic polynomial with coefficients in the subring of R generated by $\text{tr } \rho(g^{n^k})$, for any fixed $n > 1$ and varying k . We leave this to the reader. \square

(6.4) We now show the finiteness of R^{\dagger} . Apply Theorem 5.1.1 to produce a finite, totally real extension F'/F and a form f over F' such that $\bar{\rho}|_{G_{F'}}$ satisfies (A1) and (A2), $\bar{\rho}_f \cong \bar{\rho}|_{G_{F'}}$ and $t_f|_{\Sigma_{F', p}} = (t|_{F'})|_{\Sigma_{F', p}}$. By replacing F' with a finite, totally real, pre-solvable extension, and applying the same procedure as in the proof of Theorem 4.1.1 (together with Proposition 4.4.1) we may assume the above as well as the the following:

- If v' is a prime of F' lying over a prime v of F belonging to Σ then $F'_{v'}$ contains L_v .
- $\bar{\rho}|_{G_{F'}}$ is everywhere unramified.
- $\bar{\rho}|_{G_{F'}}$ is trivial at every place above Σ .
- ρ_f is admissible of type $t(v)$ at all places v above Σ .
- $\psi|_{G_{F'}} = \psi_1^2$ and $\det \rho_f = \psi_2^2 \chi_p$ with ψ_i unramified.
- The set of primes of F' lying over Σ_C has even cardinality.
- F' has even degree over \mathbb{Q} .
- k contains the eigenvalues of the image of $\bar{\rho}$.

Replace f by $\psi_2^{-1} \cdot f$ and let $\bar{\rho}_1$ be $\psi_1^{-1} \cdot \bar{\rho}|_{G_{F'}}$. We now define an deformation datum $\mathcal{D}^\circ = (t', \Sigma^{\text{ram}}, \Sigma^{\text{aux}})$ over F' (with respect to $\bar{\rho}_1$) by taking Σ^{ram} to be the set of primes lying over $\Sigma \setminus \Sigma_p$, t' to be the restriction of t to F' , Σ^{aux} to be $\{w\}$ where w is any sufficiently large prime satisfying the necessary conditions (see, for example, the proof of Theorem 4.1.1). We use the notation of §3.3 with an overline to indicate the relevant deformation rings. The above conditions fulfill the hypotheses of Theorem 3.11.1, so we conclude that $\bar{R}_{\mathcal{D}^\circ}$ is finite over \mathcal{O} . Note that tensoring by ψ_1 gives a bijection between deformations of $\bar{\rho}_1$ and deformations of $\bar{\rho}|_{G_{F'}}$, and this bijection preserves any quality which we care about. In particular, it furnishes a natural map $\bar{R}_{\Sigma(\mathcal{D}^\circ)} \rightarrow R_\Sigma$ even though the first ring classifies deformations with determinant χ_p while the second classifies those with determinant $\psi \chi_p$. The following lemma completes the proof of Theorem 6.1.1.

Lemma 6.4.1. *The map $\bar{R}_{\Sigma(\mathcal{D}^\circ)} \rightarrow R_\Sigma \rightarrow R^\dagger$ factors through $\bar{R}_{\mathcal{D}^\circ}$.*

Proof. It suffices to prove the result on framed deformation rings. For this, it suffices to show that the map

$$\bar{R}^{\square, \text{loc}} \rightarrow R^{\square, \text{loc}} \rightarrow R^{\square, \text{loc}, \dagger}$$

factors through $\bar{R}_{\mathcal{D}^\circ}^{\square, \text{loc}}$. Since $R^{\square, \text{loc}, \dagger}$ is reduced and flat over \mathcal{O} it suffices to show that if E'/E is a finite extension and $R^{\square, \text{loc}, \dagger} \rightarrow E'$ a point then the resulting map $\bar{R}^{\square, \text{loc}} \rightarrow E'$ factors through $\bar{R}_{\mathcal{D}^\circ}^{\square, \text{loc}}$. This can easily be checked place by place. \square

7. LIFTING PROBLEMS

(7.1) Let F_v/\mathbb{Q}_ℓ be a finite extension ($\ell = p$ allowed) and let E be any extension of \mathbb{Q}_p . An *inertial type* for F_v is a representation $\tau_v : I_{F_v} \rightarrow \text{GL}_2(E)$ with open kernel. We say that a representation $\rho_v : G_{F_v} \rightarrow \text{GL}_2(E)$ (assumed to be de Rham if $\ell = p$) has *inertial type* τ_v if the Weil-Deligne representation associated to ρ_v is isomorphic to τ_v when restricted to inertia. The “inertial type” knows nothing about the monodromy operator of the Weil-Deligne representation. For example, in the $\ell \neq p$ case “inertial type” does not distinguish between unramified representations and representations with unipotent inertia. Do not confuse “inertial type” and “type”: they are very different. In fact, by Proposition 2.6.1, they are complementary (at least in global situations): “type” determines the data of the monodromy operator, exactly what “inertial type” forgets.

(7.2) We now apply Theorem 6.1.1 to study certain lifting problems. Let $\bar{\rho} : G_F \rightarrow \text{GL}_2(\bar{\mathbb{F}}_p)$ be a given representation. A *lifting problem* is a tuple $\mathcal{P} = (\Sigma, \psi, t, \{\tau_v\}_{v \in \Sigma})$ consisting of:

- A finite set Σ of finite places of F , including all those at which $\bar{\rho}$ ramifies and all those over p .
- A finite order character $\psi : G_F \rightarrow \bar{\mathbb{Q}}_p^\times$, unramified outside Σ , such that $\det \bar{\rho} = \bar{\psi} \cdot \bar{\chi}_p$.
- A definite type function t defined on Σ .
- For each $v \in \Sigma$ an inertial type τ_v .

A *solution* to a lifting problem \mathcal{P} is a representation $\rho : G_F \rightarrow \text{GL}_2(\bar{\mathbb{Q}}_p)$ satisfying the following:

- ρ is a weight two lift of $\bar{\rho}$.
- $\det \rho = \psi \chi_p$.
- ρ is unramified outside of Σ .
- $\rho|_{G_{F_v}}$ has type $t(v)$ for $v \in \Sigma$.
- $\rho|_{G_{F_v}}$ has inertial type τ_v for $v \in \Sigma$.

A *local solution* to a lifting problem \mathcal{P} is a family $\{\rho_v\}_{v \in \Sigma}$ consisting of representations $\rho_v : G_{F_v} \rightarrow \text{GL}_2(\bar{\mathbb{Q}}_p)$ which satisfy the following:

- The reduction of some stable lattice in ρ_v is isomorphic to $\bar{\rho}|_{G_{F_v}}$.
- $\det \rho_v = \psi \chi_p|_{G_{F_v}}$.

- For v over p the representation ρ_v has weight two.
- ρ_v has definite type $t(v)$.
- ρ_v has inertial type τ_v .

Note that we require ρ_v to have *definite type* in the above that is, it cannot have type A/C or AB/C . The main result of §7 is the following theorem.

Theorem 7.2.1. *Let $\bar{\rho} : G_F \rightarrow \mathrm{GL}_2(\bar{\mathbb{F}}_p)$ be an odd representation satisfying (A1) and (A2) and let \mathcal{P} be a lifting problem. Then there are only finitely many solutions to \mathcal{P} . A solution exists if and only if a local solution exists.*

To prove this theorem, we will use Theorem 6.1.1. The work lies in showing that the necessary local deformation rings exist. This has basically been done already by Kisin and Gee, though we will need some variants on their work. We establish the necessary results in the next two sections. After proving Theorem 7.2.1 we apply it to give a proof of Theorem 1.1.3.

(7.3) We now establish some results about local deformation rings in the unequal characteristic case. Let F_v/\mathbb{Q}_ℓ be a finite extension with $\ell \neq p$, let $\bar{\rho}_v : G_{F_v} \rightarrow \mathrm{GL}_2(k)$ be a representation and let $\psi_v : G_{F_v} \rightarrow \mathrm{GL}_2(\mathcal{O})$ be a finite order character with $\det \bar{\rho}_v = \bar{\psi}_v \cdot \bar{\chi}_p$. Let R_v^{\square, ψ_v} denote the universal ring classifying framed deformations of $\bar{\rho}_v$ with determinant $\psi_v \chi_p$. The main result we need is the following:

Proposition 7.3.1. *Let τ_v be an inertial type for F_v and let $*$ be AB or C . There exists a quotient $R_v^{\square, \psi_v, \tau_v, *}$ of R_v^{\square, ψ_v} with the following properties:*

- (1) $R_v^{\square, \psi_v, \tau_v, *}$ is \mathcal{O} -flat, reduced and all of its components have dimension 4.
- (2) Let $x : R_v^{\square, \psi_v} \rightarrow E'$ be a map corresponding to a representation ρ . If x factors through $R_v^{\square, \psi_v, \tau_v, *}$ then ρ has type $*$ and inertial type τ_v . Conversely, if ρ has definite type $*$ and inertial type τ_v then x factors through $R_v^{\square, \tau_v, *}$.

The ring $R_v^{\square, \psi_v, \tau_v, *}$ may be zero.

The subtlety in (2) above is that if ρ has *indefinite type* (i.e., type AB/C) then we cannot conclude that x factors through $R_v^{\square, \psi_v, \tau_v, *}$. We need a few lemmas before proving the proposition.

Lemma 7.3.2. *Let τ_v be an inertial type for F_v . There exists a quotient $R_v^{\square, \psi_v, \tau_v}$ of R_v^{\square, ψ_v} with the following properties:*

- (1) $R_v^{\square, \psi_v, \tau_v}$ is \mathcal{O} -flat, reduced and all of its components have dimension 4.
- (2) Let $x : R_v^{\square, \psi_v} \rightarrow E'$ be a map corresponding to a representation ρ . Then x factors through $R_v^{\square, \psi_v, \tau_v}$ if and only if ρ has inertial type τ_v .

The ring $R_v^{\square, \psi_v, \tau_v}$ may be zero.

Proof. This is exactly [Ge2, Proposition 3.1.3]. □

We now compute the components of the ring $R_v^{\square, \psi_v, \tau_v}$ in certain cases.

Lemma 7.3.3. *Assume that $\bar{\rho}$ is trivial, that F_v contains the p th roots of unity, let τ_v be the trivial inertial type and let ψ_v be the trivial character. The ring $R_v^{\square, \psi_v, \tau_v}$ then has two components, corresponding to admissible deformations of types AB and C .*

Proof. Let ρ be a deformation of $\bar{\rho}$ to an extension E'/E . One easily sees

$$\rho \text{ is admissible} \iff \rho \text{ is semi-stable with determinant } \chi_p.$$

Here “admissible” means admissible of type AB or C and “semi-stable” means inertia acts unipotently. The direction \implies shows that if x is an E' point of R_v^{\square, ψ_v} which factors through one of the rings $R_v^{\square, *}$ of Proposition 2.5.1 then x factors through $R_v^{\square, \psi_v, \tau_v}$. We thus have a closed immersion

$$\mathrm{Spec}(R_v^{\square, *}) \rightarrow \mathrm{Spec}(R_v^{\square, \psi_v, \tau_v}).$$

Proposition 2.5.1 shows that $R_v^{\square, *}$ is integral and four dimensional and so $R_v^{\square, *}$ must be a component of $R_v^{\square, \psi_v, \tau_v}$. To show that $R_v^{\square, AB}$ and $R_v^{\square, C}$ are all the components of $R_v^{\square, \psi_v, \tau_v}$ it suffices to show that any E' -point of the latter lies on one of the former. This follows from the implication \impliedby . □

We now prove the proposition.

Proof of Proposition 7.3.1. Let F'_v be the compositum of the extensions determined by $\ker \tau_v$, $\ker \bar{\rho}_v$, $\ker \psi_v$ and the extension $F_v(\zeta_p)$. Let $\bar{R}_v^{\square, \psi_v}$ be the universal ring classifying framed deformations of $\bar{\rho}|_{G_{F'}}$ with determinant $\psi_v \chi_p = \chi_p$ and let $\bar{R}_v^{\square, \psi_v, \tau_v}$ be quotient considered in the previous lemma. We have a natural map $\text{Spec}(R_v^{\square, \psi_v, \tau_v}) \rightarrow \text{Spec}(\bar{R}_v^{\square, \psi_v, \tau_v})$. By the previous lemma, we can write $\text{Spec}(\bar{R}_v^{\square, \psi_v, \tau_v}) = X_{AB} \cup X_C$ where X_* is a components and an E' -point of $\text{Spec}(\bar{R}_v^{\square, \psi_v, \tau_v})$ lies on X_* if and only if it has type $*$. We can then take $\text{Spec}(R_v^{\square, \psi_v, \tau_v, *})$ to be the union of the components of $\text{Spec}(R_v^{\square, \psi_v, \tau_v})$ which map into X_* . As $R_v^{\square, \psi_v, \tau_v, *}$ is a union of components, it satisfies (1) in the statement of the proposition. The verification of (2) is easy and left to the reader. \square

(7.4) We now establish some results about local deformations ring in the equal characteristic case. Let F_v/\mathbb{Q}_p be a finite extension, let $\bar{\rho}_v : G_{F_v} \rightarrow \text{GL}_2(k)$ be a representation and let $\psi_v : G_{F_v} \rightarrow \mathcal{O}^\times$ be a finite order character with $\det \bar{\rho}_v = \bar{\psi}_v \cdot \bar{\chi}_p$. Let R_v^{\square, ψ_v} denote the universal ring classifying framed deformations of $\bar{\rho}_v$ with determinant $\psi_v \chi_p$. The main result we need is the following:

Proposition 7.4.1. *Let τ_v be an inertial type for F_v and let $*$ be A , B or C . There exists a quotient $R_v^{\square, \psi_v, \tau_v, *}$ of R_v^{\square, ψ_v} with the following properties:*

- (1) $R_v^{\square, \psi_v, \tau_v, *}$ is \mathcal{O} -flat, reduced and all of its components have dimension $[F_v : \mathbb{Q}_p] + 4$.
- (2) Let $x : R_v^{\square, \psi_v} \rightarrow E'$ be a map corresponding to a representation ρ . If x factors through $R_v^{\square, \psi_v, \tau_v, *}$ then ρ has type $*$ and inertial type τ_v . Conversely, if ρ has definite type $*$ and inertial type τ_v then x factors through $R_v^{\square, \psi_v, \tau_v, *}$.

The ring $R_v^{\square, \psi_v, \tau_v, *}$ may be zero.

The proof follows the same basic plan as that of Proposition 7.3.1 and so we omit some of the details. We begin with the analogue of Lemma 7.3.2.

Lemma 7.4.2. *Let τ_v be an inertial type for F_v . There exists a quotient $R_v^{\square, \psi_v, \tau_v}$ of R_v^{\square, ψ_v} with the following properties:*

- (1) $R_v^{\square, \psi_v, \tau_v}$ is \mathcal{O} -flat, reduced and all of its components have dimension $[F_v : \mathbb{Q}_p] + 4$.
- (2) Let $x : R_v^{\square, \psi_v} \rightarrow E'$ be a map corresponding to a representation ρ . Then x factors through $R_v^{\square, \psi_v, \tau_v}$ if and only if ρ is weight two and has inertial type τ_v .

The ring $R_v^{\square, \psi_v, \tau_v}$ may be zero.

Proof. This follows from [Ki4, Theorem 3.3.4]. \square

We now compute the components of $R_v^{\square, \psi_v, \tau_v}$ in certain cases, as in Lemma 7.3.3.

Lemma 7.4.3. *Assume that $\bar{\rho}$ is trivial, that F_v contains the p th roots of unity, let τ_v be the trivial inertial type and let ψ_v be the trivial character. The ring $R_v^{\square, \psi_v, \tau_v}$ then has three components, corresponding to admissible deformations of types A , B and C .*

Proof. The proof is essentially the same as that of Lemma 7.3.3. Proposition 2.2.1 shows that the space of deformations of $\bar{\rho}$ which are admissible of a given type form a component. An E' -point of $R_v^{\square, \psi_v, \tau_v}$ gives rise to a semi-stable representation with determinant χ_p reducing to the trivial representation, and any such representation is admissible. Thus the stated components are all of the components. \square

The proof of Proposition 7.4.1 now follows exactly as the proof of Proposition 7.3.1.

(7.5) We now prove the Theorem 7.2.1. A global solution of \mathcal{P} gives rise to a local solution, since we know that a global solution has definite type (Theorem 1.1.1 and Proposition 2.6.1). Thus if no local solution to \mathcal{P} exists then no solution exists. Assume now that \mathcal{P} admits a local solution. Let E/\mathbb{Q}_p be a large finite extension and let \mathcal{O} be its ring of integers. For $v \in \Sigma$ we define R_v^\dagger to be the ring $R_v^{\square, \psi_v, \tau_v, t(v)}$ constructed in the previous sections, where $\psi_v = \psi|_{G_{F_v}}$. These rings are non-zero due to the existence of a local solution. They satisfy the conditions specified in §6.1 by Proposition 7.3.1 and Proposition 7.4.1. Let R^\dagger be the ring defined in §6.1 associated to the above local deformation rings. We then see that any \mathbb{Q}_p -point of R^\dagger gives a solution to \mathcal{P} , and all solutions are of this form. Theorem 6.1.1 gives the desired result.

(7.6) We now give a simplified version of Theorem 7.2.1, which is easier to apply in many circumstances. Before doing this, we introduce some terminology. Let F_v/\mathbb{Q}_ℓ be a finite extension ($\ell = p$ allowed) and let

$\bar{\rho}_v : G_{F_v} \rightarrow \mathrm{GL}_2(\bar{\mathbb{F}}_p)$ be a representation. We say that a definite type $*$ is *compatible* with $\bar{\rho}_v$ if $\bar{\rho}_v$ admits a lift to $\bar{\mathbb{Q}}_p$ (required to be weight two if $v \mid p$) which is definite of type $*$. Similarly, for a representation $\bar{\rho} : G_F \rightarrow \mathrm{GL}_2(\bar{\mathbb{F}}_p)$ with F/\mathbb{Q} finite, we may speak of the compatibility of $\bar{\rho}$ with a definite type function. We now have the following result:

Theorem 7.6.1. *Let $\bar{\rho} : G_F \rightarrow \mathrm{GL}_2(\bar{\mathbb{F}}_p)$ be an odd representation satisfying (A1) and (A2), let $\psi : G_F \rightarrow \bar{\mathbb{Q}}_p^\times$ be a finite order character such that $\det \bar{\rho} = \bar{\psi} \cdot \bar{\chi}_p$, let Σ be a finite set of places including all those at which $\bar{\rho}$ or ψ ramify and those above p , let Σ' be a subset of Σ and let t be a definite type function on Σ' compatible with $\bar{\rho}$. Then $\bar{\rho}$ admits a weight two lift to $\bar{\mathbb{Q}}_p$ which is unramified outside of Σ , has determinant $\psi \chi_p$ and has type t at the places in Σ' .*

To prove Theorem 7.6.1 we need to know that every residual representation is compatible with some type. We will prove this (and in fact more precise results) in the following two sections. We will then return to the proof of Theorem 7.6.1.

(7.7) Let F_v/\mathbb{Q}_ℓ be a finite extension with $\ell \neq p$. The purpose of this section is to prove the following proposition:

Proposition 7.7.1. *Given a representation $\bar{\rho}_v : G_{F_v} \rightarrow \mathrm{GL}_2(\bar{\mathbb{F}}_p)$ there exists a representation $\rho_v : G_{F_v} \rightarrow \mathrm{GL}_2(\bar{\mathbb{Q}}_p)$ which lifts $\bar{\rho}_v$, has the same conductor as $\bar{\rho}_v$ and has definite type. In particular, $\bar{\rho}_v$ is compatible with some type.*

Proof. Write $\bar{\rho}$ in place of $\bar{\rho}_v$. Let $G = G_{F_v}$, let $I = I_{F_v}$ be the inertia subgroup and let I^w be the wild inertia subgroup. Let I' be the closed normal subgroup of G containing I^w and whose image in G/I^w is the prime-to- p part of tame inertia. The representation $\bar{\rho}|_{I'}$ is semi-simple since I' is prime-to- p . We consider three cases: (1) $\bar{\rho}|_{I'}$ is irreducible; (2) $\bar{\rho}|_{I'} = \bar{\alpha} \oplus \bar{\beta}$ with $\bar{\alpha} \neq \bar{\beta}$; and (3) $\bar{\rho}|_{I'} = \bar{\alpha} \oplus \bar{\alpha}$. We first show that a definite type lift exists in each of these cases and then refine this result to produce a definite type lift with equal conductor.

Case 1. As I' has no mod p cohomology, the Hochschild-Serre spectral sequence gives $H^2(G, \mathrm{ad}^\circ \bar{\rho}) = H^2(G/I', (\mathrm{ad}^\circ \bar{\rho})^{I'})$. The latter group vanishes since $(\mathrm{ad}^\circ \bar{\rho})^{I'} = 0$. Thus there is no obstruction to deforming $\bar{\rho}$ and so we can lift it to $\bar{\mathbb{Q}}_p$. Furthermore, we find a lift whose determinant is any prescribed lift of the determinant of $\bar{\rho}$. We can thus find a lift ρ whose determinant is a finite order character times the cyclotomic character. Since ρ is also irreducible it follows that ρ is definite of type AB .

Case 2. The group G must permute the characters $\bar{\alpha}$ and $\bar{\beta}$. First assume that G permutes them trivially. Then both extend to characters of G and $\bar{\rho} = \bar{\alpha} \oplus \bar{\beta}$. Pick characters $\alpha, \beta, \gamma, \delta$ of G taking values in $\bar{\mathbb{Q}}_p^\times$ as follows: α and β are finite order and lift $\bar{\alpha}$ and $\bar{\beta}$; γ is unramified of infinite order and with trivial reduction; and δ is a finite order character reducing to the cyclotomic character. Then $\rho = (\alpha\gamma\delta^{-1}\chi_p) \oplus (\beta\gamma^{-1})$ is a lift of $\bar{\rho}$ which is definite of type AB .

Now assume that G permutes α and β non-trivially. Let H be the subgroup of G which stabilizes $\bar{\alpha}$ and $\bar{\beta}$. Then $\bar{\alpha}$ and $\bar{\beta}$ extend to characters of H and $\bar{\rho}$ is the induction of either of them from H to G . Let $\alpha : H \rightarrow \bar{\mathbb{Q}}_p^\times$ be a finite order lift of $\bar{\alpha}$ and let $\gamma : G \rightarrow \bar{\mathbb{Q}}_p^\times$ be a character which reduces to the trivial character and is such that γ^2 is a finite order character times the cyclotomic character. Then $\rho = \mathrm{Ind}_H^G(\alpha\gamma)$ is a definite type AB representation lifting $\bar{\rho}$. (It does not have type C since it is irreducible.)

Case 3. The character $\bar{\alpha}$ is fixed by G and thus extends to G . We can thus twist $\bar{\rho}$ by $\bar{\alpha}^{-1}$ and assume that $\bar{\rho}(I') = 1$. We can therefore regard $\bar{\rho}$ as a representation of G/I' . This group is topologically generated by F and N subject to the relation $FN F^{-1} = N^q$, where q is the cardinality of the residue field of F_v , and the topological condition that N is pro- p . If $\bar{\rho}(N) = 1$ then $\bar{\rho}$ is unramified and it is easy to produce a definite lift. Thus assume $N \neq 0$. We can then pick a basis so that

$$\bar{\rho}(N) = \begin{pmatrix} 1 & * \\ & 1 \end{pmatrix}$$

It follows from the relation $FN F^{-1} = N^q$ that

$$\bar{\rho}(F) = \begin{pmatrix} qa & * \\ & a \end{pmatrix}.$$

We can twist $\bar{\rho}$ by an unramified character so that $a = 1$. We then find that $\bar{\rho}$ is a non-trivial extension of 1 by $\bar{\chi}_p$. Say that $\bar{\rho}$ is defined over the finite field k and let \mathcal{O} be the Witt vectors of k . Consider the map

$$H^1(G_{F_v}, \mathcal{O}(\chi_p)) \rightarrow H^1(G_{F_v}, k(\chi_p)).$$

It is surjective by Kummer theory; in fact, every element in the target is the image of a non-torsion element of the source. The representation $\bar{\rho}$ is represented by a class \bar{c} in the target. Let c be a non-torsion class in the source lifting \bar{c} . Then the image of c in $H^1(G_{F_v}, E(\chi_p))$ is non-zero and corresponds to a definite type C lift of $\bar{\rho}$.

Controlling the conductor. We now show how one can refine the above results to produce definite type lifts of $\bar{\rho}$ with the same conductor as $\bar{\rho}$. Observe that if ρ is a lift of $\bar{\rho}$ then

$$f(\bar{\rho}) - f(\rho) = \dim(\bar{\rho}^I) - \dim(\rho^I) \geq 0$$

where f is the exponent of the conductor. Now, if $\dim(\bar{\rho}^I) = 2$ then $\bar{\rho}$ is unramified. It is not hard to produce a definite type AB lift ρ which is unramified, which shows that one can find a definite type lift of $\bar{\rho}$ with the same conductor.

Now consider the case where $\dim(\bar{\rho}^I) = 1$. We then have that $\bar{\rho}^I$ is either one or two dimensional. First consider the case where $\dim(\bar{\rho}^I) = 1$. Then $\bar{\rho}|_{I'} = \bar{\alpha} \oplus 1$ where $\bar{\alpha}$ is non-trivial. As in Case 2 we find that $\bar{\alpha}$ extends to G and $\bar{\rho} = \bar{\alpha} \oplus 1$. The definite type AB lift ρ produced in Case 2 (using $\beta = 1$) has $\dim(\bar{\rho}^I) = 1$ and thus has the same conductor as $\bar{\rho}$. Now consider the case where $\dim(\bar{\rho}^I) = 2$ so that $\bar{\rho}(I') = 1$. We then find that $\bar{\rho}|_I$ is a non-trivial extension of 1 by $\bar{\chi}_p$. The definite type C lift ρ produced in Case 3 above is such that $\rho|_I$ is a non-trivial extension of 1 by χ_p . We thus find $\dim(\rho^I) = 1$ which shows that ρ and $\bar{\rho}$ have the same conductor. This completes the proof. \square

(7.8) Now let F_v/\mathbb{Q}_p be a finite extension. The purpose of this section is to prove the following proposition:

Proposition 7.8.1. *Given a representation $\bar{\rho}_v : G_{F_v} \rightarrow \mathrm{GL}_2(\bar{\mathbb{F}}_p)$ there exists a representation $\rho_v : G_{F_v} \rightarrow \mathrm{GL}_2(\bar{\mathbb{Q}}_p)$ which lifts $\bar{\rho}_v$, has weight two and has definite type. In particular, $\bar{\rho}_v$ is compatible with some type.*

We first need a lemma.

Lemma 7.8.2. *Let F'_v be a quadratic extension of F_v . Then there exists a character $\psi : G_{F'_v} \rightarrow \bar{\mathbb{Q}}_p^\times$ with the following two properties: (1) ψ and ψ' are de Rham with non-positive Hodge-Tate weights; and (2) $\psi \cdot \psi'$ is a finite order character times the cyclotomic character. Here ψ' is the conjugate of ψ by $\mathrm{Gal}(F'_v/F_v)$.*

Proof. Let S be the set of all embeddings of F'_v into $\bar{\mathbb{Q}}_p$. Each element of $i \in S$ yields, via class field theory, a character $\gamma_i : W_{F'_v} \rightarrow \bar{\mathbb{Q}}_p^\times$ where $W_{F'_v}$ is the Weil group of F'_v . These characters are de Rham with non-positive Hodge-Tate weights. Pick a subset S_0 of S such that S is the disjoint union of S_0 and σS_0 , where σ is the non-trivial element of $\mathrm{Gal}(F'_v/F_v)$. Let ϕ be the product of the γ_i with $i \in S_0$. The conjugate ϕ' is then the product of the γ_i with $i \in \sigma S_0$. The characters ϕ and ϕ' are de Rham with non-positive Hodge-Tate weights since each γ_i is. Furthermore, $\phi \cdot \phi'$ corresponds via class field theory to the composite $(F'_v)^\times \rightarrow \mathbb{Q}_p^\times \rightarrow \bar{\mathbb{Q}}_p^\times$, where the first map is the norm map and the second is the canonical inclusion. We thus see that $\phi \cdot \phi'$ agrees with the cyclotomic character on inertia. It follows that we can take ψ to be ϕ times an appropriate unramified character. \square

We now return to the proof of the proposition.

Proof of Proposition 7.8.1. Write G in place of G_{F_v} . We first consider the case where $\bar{\rho} = \bar{\rho}_v$ is reducible. Say that $\bar{\rho}$ is defined over the finite field k and let \mathcal{O} be the Witt vectors of k . By twisting we can assume $\bar{\rho}$ has the form

$$\begin{pmatrix} \bar{\alpha} \cdot \bar{\chi}_p & * \\ & 1 \end{pmatrix}.$$

Thus $\bar{\rho}$ corresponds to a class \bar{c} of $H^1(G, k(\bar{\alpha} \cdot \bar{\chi}_p))$. If $\bar{\alpha} = 1$ then, as was the case in the proof of Proposition 7.7.1 we can find a non-torsion class c in $H^1(G, \mathcal{O}(\chi_p))$ lifting the class \bar{c} . The representation corresponding to c is then a definite type C lift of $\bar{\rho}$. Now say $\bar{\alpha} \neq 1$. Let α be a finite order lift of $\bar{\alpha}$ and let $\gamma : G \rightarrow \mathcal{O}^\times$ be an infinite order unramified character reducing to the trivial character. We have an exact sequence

$$0 \rightarrow (\mathcal{O}/\mathfrak{m})(\gamma^2\alpha) \rightarrow (\mathcal{O}/\mathfrak{m}^{n+1})(\gamma^2\alpha) \rightarrow (\mathcal{O}/\mathfrak{m}^n)(\gamma^2\alpha) \rightarrow 0$$

which yields

$$H^1(G, (\mathcal{O}/\mathfrak{m}^{n+1})(\gamma^2 \alpha \chi_p)) \rightarrow H^1(G, (\mathcal{O}/\mathfrak{m}^n)(\gamma^2 \alpha \chi_p)) \rightarrow H^2(G, (\mathcal{O}/\mathfrak{m})(\gamma^2 \alpha \chi_p)).$$

Now, $(\mathcal{O}/\mathfrak{m})(\gamma^2 \alpha \chi_p)$ is just $k(\bar{\alpha} \cdot \bar{\chi}_p)$. The group $H^2(G, k(\bar{\alpha} \cdot \bar{\chi}_p))$ is dual to $H^0(G, k(\bar{\alpha}^{-1}))$ which vanishes since $\bar{\alpha} \neq 1$. We thus see that the map on H^1 groups above is surjective. It follows that the map

$$H^1(G, \mathcal{O}(\gamma^2 \alpha \chi_p)) \rightarrow H^1(G, k(\bar{\alpha} \cdot \bar{\chi}_p))$$

is surjective. Let c be any class lifting \bar{c} and let ρ' be the corresponding representation. Then $\rho = \gamma^{-1} \otimes \rho'$ is a definite type A lift of $\bar{\rho}$.

We now consider the case where $\bar{\rho}$ is irreducible. Let I^w be the wild inertia subgroup of G . As I^w is pro- p the space of invariants $\bar{\rho}^{I^w}$ is non-zero. Since I^w is normal, this space is stable under G . It follows that it must be all of $\bar{\rho}$. We thus see that $\bar{\rho}(I^w) = 1$, and so $\bar{\rho}$ may be regarded as a representation of G/I^w . As the tame inertia group I^t is prime-to- p , we have $\bar{\rho}|_{I^t} = \bar{\alpha} \oplus \bar{\beta}$. If $\bar{\alpha} = \bar{\beta}$ then $\bar{\alpha}$ extends to G and after twisting by $\bar{\alpha}^{-1}$ the representation $\bar{\rho}$ would be unramified, and therefore not irreducible. Thus $\bar{\alpha} \neq \bar{\beta}$. Now, G permutes the two characters $\bar{\alpha}$ and $\bar{\beta}$. If it permuted them trivially then each would extend to all of G and $\bar{\rho}$ would be reducible. We thus see that G permutes $\bar{\alpha}$ and $\bar{\beta}$ non-trivially. Let H be the subgroup of G which fixes $\bar{\alpha}$. The character $\bar{\alpha}$ extends to H and $\bar{\rho}$ is the induction of $\bar{\alpha}$ from H to G . Let ψ be the character of $G_{F_v'} \rightarrow \overline{\mathbb{Q}_p}^\times$ afforded by Lemma 7.8.2. Let $\gamma : G_{F_v'} \rightarrow \overline{\mathbb{Q}_p}^\times$ be a finite order character lifting the residual character $\bar{\alpha} \cdot \bar{\psi}^{-1}$. Put $\rho = \text{Ind}_H^G(\gamma\psi)$. It is clear that ρ is a lift of $\bar{\rho}$. Now, $\rho|_H = (\gamma\psi) \oplus (\gamma'\psi')$ which shows that ρ is de Rham with non-positive Hodge-Tate weights. As $\det \rho$ is a finite order character times the cyclotomic character, it follows that ρ is weight two. Since $\bar{\rho}$ is irreducible, so is ρ , which shows that ρ is definite of type B . \square

Remark 7.8.3. Proposition 7.7.1 and Proposition 7.8.1 show that any residual representation is compatible with some type. Note, however, that it is not always possible to find a lift of a specified type. For instance, residual representations which are irreducible will not admit lifts of type C . When $v \nmid p$ it is not difficult to determine exactly which types a given residual representation are compatible with. When $v \mid p$ we have not worked this out completely.

(7.9) We now prove Theorem 7.6.1. By Proposition 7.7.1 and Proposition 7.8.1 we can extend t to a type function on all of Σ which is compatible with $\bar{\rho}$. We may thus assume $\Sigma = \Sigma'$. For each $v \in \Sigma$ let ρ'_v be a lift of $\bar{\rho}|_{G_{F_v}}$ which is weight two for $v \mid p$ and has definite type $t(v)$. The existence of these lifts is assured by the compatibility of $\bar{\rho}$ and t . The character $\psi\chi_p \cdot (\det \rho'_v)^{-1}$ is finite and pro- p . Let ρ_v be the twist of ρ'_v by the square root of this character. Then ρ_v is a lift of $\bar{\rho}|_{G_{F_v}}$, has weight two for $v \mid p$, has definite type $t(v)$ and also has determinant $\psi\chi_p|_{G_{F_v}}$. Let τ_v be the inertial type of ρ_v . We have thus produced a locally solvable lifting problem $\mathcal{P} = (\Sigma, \psi, t, \{\tau_v\})$. Theorem 7.2.1 shows that \mathcal{P} has a solution, which gives the required lift.

(7.10) We close §7 by establishing the first statement of Theorem 1.1.3. Precisely, we prove the following (see also Remark 9.4.5):

Proposition 7.10.1. *Let $\bar{\rho} : G_F \rightarrow \text{GL}_2(\overline{\mathbb{F}_p})$ be an irreducible odd representation satisfying (A1) and (A2). Then there is a finitely ramified, weight two lift $\rho : G_F \rightarrow \text{GL}_2(\overline{\mathbb{Q}_p})$ of $\bar{\rho}$ such that the prime-to- p conductors of ρ and $\bar{\rho}$ agree.*

Proof. Write $\det \bar{\rho} = \bar{\psi} \cdot \bar{\chi}_p$ and let ψ be the Teichmüller lift of $\bar{\psi}$. Let Σ be the union of the set of primes above p and the set of primes at which $\bar{\rho}$ ramifies. For each $v \nmid p$ in Σ use Proposition 7.7.1 to produce a definite type lift $\rho'_v : G_{F_v} \rightarrow \text{GL}_2(\overline{\mathbb{Q}_p})$ of $\bar{\rho}|_{G_{F_v}}$ such that ρ'_v and $\bar{\rho}_v$ have the same conductor. Let ρ_v be the twist of ρ'_v by the square root of $\psi\chi_p \cdot (\det \rho'_v)^{-1}$, so that ρ_v has determinant $\psi\chi_p$. One easily sees that ρ_v has the same conductor as ρ'_v . For $v \mid p$ use Proposition 7.8.1 to produce a definite type weight two lift ρ'_v of $\bar{\rho}|_{G_{F_v}}$ and let ρ_v be an appropriate twist with determinant $\psi\chi_p$. For $v \in \Sigma$ let $t(v)$ be the type of ρ_v and let τ_v be the inertial type of ρ_v . We have thus defined a lifting problem $\mathcal{P} = (\Sigma, \psi, t, \{\tau_v\})$ which is locally solvable. Let ρ be a solution to $\bar{\rho}$. For $v \nmid p$ in Σ the representations $\rho|_{G_{F_v}}$ and ρ_v have the same type and inertial type and so $\rho|_{I_{F_v}} \cong \rho_v|_{I_{F_v}}$. This shows that ρ has the same prime-to- p conductor as $\bar{\rho}$. \square

8. TWO ADDITIONAL RESULTS

(8.1) We now give two additional results, building on the main theorems we have proved. The first could be called “solvable descent for mod p Hilbert eigenforms.” Recall that Langlands’ base change implies that if ρ is a p -adic representation of G_F and F'/F is a solvable extension then ρ is modular if and only if $\rho|_{G_{F'}}$ is. If $\bar{\rho}$ is a residual representation of G_F then the modularity of $\bar{\rho}$ clearly implies that of $\bar{\rho}|_{G_{F'}}$ by the previous sentence. However, the other direction — descent — is not so clear. Khare [Kh] has established some results in the case $F = \mathbb{Q}$. We prove the following:

Proposition 8.1.1. *Let $\bar{\rho} : G_F \rightarrow \mathrm{GL}_2(\overline{\mathbb{F}}_p)$ be a representation satisfying (A1) and (A2) and let $t : \Sigma_p \rightarrow \{A, B, C\}$ be a type function compatible with $\bar{\rho}$. Assume there exists a finite, solvable, totally real extension F'/F and a parallel weight two Hilbert eigenform f over F' with coefficients in $\overline{\mathbb{Q}}_p$ such that $\bar{\rho}|_{G_{F'}}$ still satisfies (A1) and (A2) and we have $\bar{\rho}|_{G_{F'}} \cong \bar{\rho}_f$ and $t_f|_{\Sigma_p} = t|_{F'}$. Then there exists a parallel weight two Hilbert eigenform g over F with coefficients in $\overline{\mathbb{Q}}_p$ such that $\bar{\rho} \cong \bar{\rho}_g$ and $t_g = t$.*

Proof. By Theorem 7.6.1 we can find a lift ρ of $\bar{\rho}$ such that $t_\rho|_{\Sigma_p} = t$. Then $\rho|_{G_{F'}}$ and ρ_f have the same type above p and isomorphic reduction and so Theorem 4.4.2 implies the modularity of $\rho|_{G_{F'}}$. Langlands’ base change now implies the modularity of ρ and thus of $\bar{\rho}$. \square

(8.2) Our second result is a strengthening of our potential modularity theorem.

Proposition 8.2.1. *Let S be a finite set of places of F . Then in Theorem 5.1.2 one can assume that the extension F'/F splits at all places in S . The same conclusion holds in Theorem 5.1.1 under the additional hypothesis that for $v \in S \cap \Sigma_p$ the type $t(v)$ is compatible with $\bar{\rho}|_{G_{F_v}}$.*

To prove this we use a combination of solvable descent and the following modification of the theorem of Moret-Bailly.

Proposition 8.2.2. *Let $(X, \Sigma, \{L_v\}, \{\Omega_v\})$ be a Skolem datum (see §5.2) and let S be a finite set of places of F . One can then find a finite solvable extension F_1/F which splits over each L_v , a finite extension F_2/F which splits at all places in S and over each L_v and is linearly disjoint from F_1 and a point $x \in X(F_1F_2)$ such that for each embedding $F_1F_2 \rightarrow L_v$ the image of x in $X(L_v)$ belongs to Ω_v .*

Proof. We are free, of course, to enlarge S . We thus assume that S contains Σ and write $S = \Sigma \cup \Sigma'$. Let F_1/F be any finite, solvable extension such that $F_{1,w} = L_v$ for w lying over $v \in \Sigma$ and $X(F_{1,w})$ is non-empty for w lying over $v \in \Sigma'$. Let X' be the restriction of scalars from F_1 to F of X_{F_1} . This is smooth and geometrically connected since its base change to \overline{F} is isomorphic to a product of copies of $X_{\overline{F}}$. We have $X'(F_v) = \prod_{w|v} X(F_{1,w})$ for any place v of F . We define a Skolem datum $(X', S, \{L'_v\}, \{\Omega'_v\})$ by taking $L'_v = F_v$ for $v \in S$ and taking Ω'_v be the product of the Ω_v ’s for $v \in \Sigma$ and all of $X'(F_v)$ for $v \in \Sigma'$. By Proposition 5.2.2 we can find an extension F_2/F which is linearly disjoint from F_1 and which splits at S and a point $x \in X'(F_2)$ such that for each $v \in S$ and each embedding $F_2 \rightarrow F_v$ the image of x in $X'(F_v)$ lands in Ω'_v . Since F_2 is linearly disjoint from F_1 we have $X'(F_2) = X(F_1F_2)$ and so x can be regarded as a point in the latter set. The proposition easily follows. \square

We now go back to the proof of Proposition 8.2.1.

Proof of Proposition 8.2.1. We only sketch a proof. The idea is to use the above proposition in place of the original theorem of Moret-Bailly to get an abelian variety A/F_1F_2 with $A[\mathfrak{p}] = \bar{\rho}|_{G_{F_1F_2}}$ and $A[l] = \bar{\rho}'|_{G_{F_1F_2}}$. This gives the modularity of $\bar{\rho}$ over F_1F_2 as in the proof of Proposition 5.1.1. One then uses solvable descent with respect to the extension F_1F_2/F_2 to conclude that $\bar{\rho}$ is modular over F_2 . (The field F' is then just F_2 .) \square

9. CONSEQUENCES OF POTENTIAL MODULARITY

(9.1) We now establish the four statements in Corollary 1.1.2. As remarked in the introduction, many of these proofs are well-known or exist already in the literature; we feel, though, that it would be useful to have them in one place. We begin with the first statement.

Proposition 9.1.1. *Let $\rho : G_F \rightarrow \mathrm{GL}_2(\overline{\mathbb{Q}}_p)$ be a finitely ramified, odd, weight two representation such that $\overline{\rho}$ satisfies (A1) and (A2). If $v \nmid p$ is a place of F at which ρ is unramified then the eigenvalues of $\rho(\mathrm{Frob}_v)$ belong to $\overline{\mathbb{Q}} \subset \overline{\mathbb{Q}}_p$ and under any embedding into \mathbb{C} have modulus $(\mathbf{N}v)^{1/2}$.*

Proof. It is easy to see that if there exists a finite extension F'/F such that $\rho|_{G_{F'}}$ satisfies the conclusion of the proposition then so does ρ . By Theorem 5.1.2 it therefore suffices to prove the theorem when ρ is associated to a Hilbert eigenform. This has been established by Blasius [Bl], extending the earlier work of Brylinski and Labesse [BL]. \square

(9.2) We now prove statement (2) of Corollary 1.1.2. We learned this proof from a lecture given by Taylor at the Summer School on Serre's Conjecture held at Luminy in 2007. Taylor attributed the proof to Dieulefait; a sketch of the argument can be found in [Di, §3.2].

Proposition 9.2.1. *Let $\rho : G_F \rightarrow \mathrm{GL}_2(\overline{\mathbb{Q}}_p)$ be a finitely ramified, odd, weight two representation such that $\overline{\rho}$ satisfies (A1) and (A2). Then there exists a number field K , a place $v_0 \mid p$ of K , an embedding $K_{v_0} \rightarrow \overline{\mathbb{Q}}_p$ and a compatible system $\{\rho_v : G_F \rightarrow \mathrm{GL}_2(K_v)\}$ indexed by the places of K such that ρ is equivalent to $\rho_{v_0} \otimes_{K_{v_0}} \overline{\mathbb{Q}}_p$. Each representation ρ_v is finitely ramified, odd, weight two and absolutely irreducible.*

Proof. Apply Theorem 5.1.1 to produce a finite Galois totally real extension F'/F linearly disjoint from $\ker \overline{\rho}$ and a parallel weight two Hilbert eigenform f over F' with coefficients in $\overline{\mathbb{Q}}_p$ such that $\rho|_{G_{F'}} = \rho_f$. Let I be the set of fields F'' which are intermediate to F' and F and for which $\mathrm{Gal}(F'/F'')$ is solvable. For $i \in I$ we write F_i for the corresponding field. For each i we can use solvable descent to find a parallel weight two cuspidal Hilbert eigenform f_i with coefficients in $\overline{\mathbb{Q}}_p$ such that $\rho|_{G_{F_i}} = \rho_{f_i}$. Let K_i denote the field of coefficients of f_i ; note that this comes with a given embedding $K_i \rightarrow \overline{\mathbb{Q}}_p$. Let K be a number field which is Galois over \mathbb{Q} , into which each K_i embeds and which contains all roots of unity of order $[F' : F]$. Fix an embedding $K \rightarrow \overline{\mathbb{Q}}_p$ and embeddings $K_i \rightarrow K$ such that the composite $K_i \rightarrow K \rightarrow \overline{\mathbb{Q}}_p$ is the given embedding. Let v_0 be the place of K determined by the embedding $K \rightarrow \overline{\mathbb{Q}}_p$. For each place v of K and each $i \in I$ we have a representation $r_{i,v} : G_{F_i} \rightarrow \mathrm{GL}_2(K_v)$ associated to the Hilbert form f_i . It is absolutely irreducible. Note that after composing r_{i,v_0} with the embedding $\mathrm{GL}_2(K_{v_0}) \rightarrow \mathrm{GL}_2(\overline{\mathbb{Q}}_p)$ we obtain $\rho|_{G_{F_i}}$.

By Brauer's theorem, we can write

$$1 = \sum_{i \in I} n_i \mathrm{Ind}_{\mathrm{Gal}(F'/F_i)}^{\mathrm{Gal}(F'/F)}(\chi_i)$$

where the n_i are integers (possibly negative) and the χ_i are characters of $\mathrm{Gal}(F'/F_i)$ valued in K^\times . (Here we use the fact that K contains all roots of unity of order $[F' : F]$.) This equality is taken in the Grothendieck group of representations of $\mathrm{Gal}(F'/F)$ over K . Note that by taking the dimension of each side we find $\sum n_i [F_i : F] = 1$.

Let v be a place of K . For a number field M write $\mathcal{C}_{M,v}$ for the category of semi-simple continuous representations of G_M on finite dimensional K_v -vector spaces. The category $\mathcal{C}_{M,v}$ is a semi-simple abelian category. We let $K(\mathcal{C}_{M,v})$ be its Grothendieck group. It is the free abelian category on the set of irreducible continuous representations of G_M on K_v -vector spaces. We let $(,)$ be the integer valued pairing on $K(\mathcal{C}_{M,v})$ given by $(A, B) = \dim_{K_v} \mathrm{Hom}(A, B)$. This is well-defined because $\mathcal{C}_{M,v}$ is semi-simple. It is symmetric. If M'/M is a finite extension then we have adjoint functors $\mathrm{Ind}_{M'}^M : \mathcal{C}_{M',v} \rightarrow \mathcal{C}_{M,v}$ and $\mathrm{Res}_{M'}^M : \mathcal{C}_{M,v} \rightarrow \mathcal{C}_{M',v}$. (One must check, of course, that induction and restriction preserve semi-simplicity — we leave this to the reader.) These functors induce maps on the K -groups which are adjoint with respect to $(,)$. If M_1 and M_2 are two extensions of M and r_1 belongs to $\mathcal{C}_{M_1,v}$ and r_2 belongs to $\mathcal{C}_{M_2,v}$ then we have the formula

$$(2) \quad (\mathrm{Ind}_{M_1}^M(r_1), \mathrm{Ind}_{M_2}^M(r_2)) = \sum_{g \in S} (\mathrm{Res}_{M_1^g M_2}^{M_1^g}(r_1^g), \mathrm{Res}_{M_1^g M_2}^{M_2^g}(r_2^g))$$

where S is a set of representatives for $G_{M_1} \backslash G_M / G_{M_2}$, M_1^g is the field determined by $gG_{M_1}g^{-1}$ and r_1^g is the representation of $gG_{M_1}g^{-1}$ given by $x \mapsto r_1(g^{-1}xg)$. This formula is gotten by using Frobenius reciprocity and Mackey's formula.

Define

$$\rho_v = \sum_{i \in I} n_i \mathrm{Ind}_{F_i}^F(r_{i,v} \otimes \chi_i),$$

which is regarded as an element of $K(\mathcal{C}_{F,v})$. We now show that each ρ_v is (the class of) an absolutely irreducible two dimensional representation. To begin with, we have

$$\begin{aligned} \rho_{v_0} \otimes_{K_{v_0}} \overline{\mathbb{Q}_p} &= \sum_{i \in I} n_i \text{Ind}_{F_i}^F ((r_{i,v_0} \otimes_{K_{v_0}} \overline{\mathbb{Q}_p}) \otimes_K \chi_i) \\ &= \sum_{i \in I} n_i \text{Ind}_{F_i}^F ((\rho|_{F_i}) \otimes_K \chi_i) \\ &= \left[\sum_{i \in I} n_i \text{Ind}_{F_i}^F (\chi_i) \right] \otimes_K \rho \\ &= \rho \end{aligned}$$

This shows that ρ_{v_0} is (the class of) an absolutely irreducible representation.

Now let v be an arbitrary finite place of K . We have

$$\begin{aligned} (\rho_v, \rho_v) &= \sum_{i,j \in I} n_i n_j (\text{Ind}_{F_i}^F (r_{i,v} \otimes \chi_i), \text{Ind}_{F_j}^F (r_{j,v} \otimes \chi_j)) \\ &= \sum_{i,j \in I} \sum_{g \in S_{ij}} n_i n_j (\text{Res}_{F_i^g F_j}^{F_i^g} ((r_{i,v} \otimes \chi_i)^g), \text{Res}_{F_i^g F_j}^{F_j} (r_{j,v} \otimes \chi_j)) \end{aligned}$$

where we have used (2). Here S_{ij} is a set of representatives for $G_{F_1} \backslash G_F / G_{F_2}$. The representation $r_{i,v}|_{F'}$ is the representation coming from the cusp form f' and so is absolutely irreducible. It follows that the restriction of $r_{i,v}$ to any subfield of F' is absolutely irreducible. Thus the representations occurring in the pairing in the second line above are irreducible. It follows that the pairing is then either 1 or 0 if the representations are isomorphic or not. Therefore, if let $\delta_{v,i,j,g}$ be 1 or 0 according to whether $\text{Res}_{F_i^g F_2}^{F_i^g} (r_{i,v} \otimes \chi_i)^g$ is isomorphic to $\text{Res}_{F_i^g F_2}^{F_j} (r_{j,v} \otimes \chi_j)$ then we find

$$(\rho_v, \rho_v) = \sum_{i,j \in I} \sum_{g \in S_{ij}} n_i n_j \delta_{v,i,j,g}.$$

Now, the $\{r_{i,v}\}_v$ and the $\{r_{j,v}\}_v$ form a compatible system. It follows that $\delta_{v,i,j,g}$ is independent of v . The above formula thus gives

$$(\rho_v, \rho_v) = (\rho_{v'}, \rho_{v'})$$

if v' is another place of K . Taking $v' = v_0$ and using that ρ_{v_0} is an absolutely irreducible representation gives $(\rho_v, \rho_v) = 1$. Now, if we write $\rho_v = \sum m_i \pi_i$ where $m_i \in \mathbb{Z}$ and the π_i are mutually non-isomorphic irreducible representations then we have $(\rho_v, \rho_v) = \sum m_i^2 (\pi_i, \pi_i)$. Since the terms are all non-negative integers and the sum is 1, we find $\rho_v = \pm \pi$ with $(\pi, \pi) = 1$. Thus π is an absolutely irreducible representation. Now, $\dim \rho_v = 2$ since each $r_{i,v}$ is two dimensional and $\sum n_i [F_i : F] = 1$. Since $\dim \pi$ is non-negative, we must have $\rho_v = \pi$. This proves that ρ_v is the class of an absolutely irreducible representation.

We must now show that ρ_v is finitely ramified, odd, weight two and compatible with ρ_{v_0} . That ρ_v is finitely ramified follows immediately from the definition. It also follows immediately from the definition that $\rho_v|_{F'}$ is equivalent to $\rho_{f',v}$. This shows that ρ_v is odd and weight two. To show that ρ_v is compatible with ρ_{v_0} look at the traces of Frobenii in ρ_v ; we leave the details to the reader. \square

Remark 9.2.2. The compatible system constructed above is in fact *strongly compatible*. For a discussion of this, see [Tay, Theorem 6.6].

(9.3) We now discuss statement (3) of the corollary. We prefer not to give a formal proof so as to avoid giving formal definitions for L -functions and their functional equations. For details, see [Tay3, Corollary 2.2] or [Tay, Theorem 6.6]: the following sketch is simply a paraphrasing of the arguments there. Let $\rho : G_F \rightarrow \text{GL}_2(\mathbb{Q}_p)$ be given. Pick an extension F'/F such that $\rho|_{G_{F'}}$ is modular. Using notation as in the previous section, write

$$1 = \sum_{i \in I} n_i \text{Ind}_{F_i}^F (\chi_i)$$

so that

$$\rho = \sum_{i \in I} n_i \text{Ind}_{F_i}^F (r_i \otimes \chi_i).$$

Taking L -functions gives

$$L(s, \rho) = \prod_{i \in I} L(r_i \otimes \chi_i, s)^{n_i}.$$

Since the representations r_i are modular, the L -functions on the right side have meromorphic continuation and functional equations. It follows that the same holds for the L -function on the left side. We also claimed in (3) that $L(s, \rho)$ converges for $\operatorname{Re} s > \frac{3}{2}$ (by which we mean, of course, the Euler product defining $L(s, \rho)$ converges in this region); this follows immediately from the bounds given in (1).

(9.4) We now turn to claim (4) of the corollary. We follow [Tay3, Corollary 2.4] in our approach. The precise statements we prove are the following:

Proposition 9.4.1. *Let $\rho : G_F \rightarrow \operatorname{GL}_2(\overline{\mathbb{Q}}_p)$ be a finitely ramified, odd, weight two representation such that $\bar{\rho}$ satisfies (A1) and (A2). Assume furthermore that F has odd degree or that there exists a place $v \nmid p$ for which $\rho|_{G_{F_v}}$ is indecomposable. Then there exists a number field K , an embedding $K \rightarrow \overline{\mathbb{Q}}_p$ and a $\operatorname{GL}_2(K)$ -type abelian variety A/F such that ρ is equivalent to $T_p A \otimes_{\mathcal{O}_K} \overline{\mathbb{Q}}_p$.*

Proposition 9.4.2. *Let $\bar{\rho} : G_F \rightarrow \operatorname{GL}_2(\overline{\mathbb{F}}_p)$ be an irreducible odd representation. If $p = 5$ and the projective image of $\bar{\rho}$ is $\operatorname{PGL}_2(\mathbb{F}_5)$ then assume $[F(\zeta_p) : F] = 4$. Then there exists a number field K , a prime \mathfrak{p} of K lying above p and a $\operatorname{GL}_2(K)$ -type abelian variety A/F such that $\bar{\rho}$ is equivalent to $A[\mathfrak{p}]$.*

We prove Proposition 9.4.1 by combining the following statements: (1) ρ is potentially modular; (2) the conclusion of the proposition is valid for modular representations (see Lemma 9.4.3 below); and (3) the conclusion of the proposition can be checked potentially (see Lemma 9.4.4 below). We prove Proposition 9.4.2 by using our previous results to produce an appropriate lift of the residual representation and then applying Proposition 9.4.1 (although a separate argument is used when (A1) does not hold). Note that Proposition 9.4.2 gives statement (2) of Theorem 1.1.3.

In what follows, we say that a representation $\rho : G_F \rightarrow \operatorname{GL}_2(\overline{\mathbb{Q}}_p)$ (resp. a representation $\bar{\rho} : G_F \rightarrow \operatorname{GL}_2(\overline{\mathbb{F}}_p)$) comes from a GL_2 -type abelian variety if there is a number field K , an embedding $K \rightarrow \overline{\mathbb{Q}}_p$ and a $\operatorname{GL}_2(K)$ -type abelian variety A/F such that ρ is equivalent to $T_p A \otimes_{\mathcal{O}_K} \overline{\mathbb{Q}}_p$ (resp. such that $\bar{\rho}$ is equivalent to $A[\mathfrak{p}] \otimes_{k_{\mathfrak{p}}} \overline{\mathbb{F}}_p$, where \mathfrak{p} is the prime determined by $K \rightarrow \overline{\mathbb{Q}}_p$, $k_{\mathfrak{p}}$ is its residue field and $k_{\mathfrak{p}} \rightarrow \overline{\mathbb{F}}_p$ is the embedding determined by $K \rightarrow \overline{\mathbb{Q}}_p$). The following lemma says that modular representations often satisfy this condition:

Lemma 9.4.3. *Let F be a totally real field and let f be a parallel weight two cuspidal Hilbert eigenform over F with coefficients in $\overline{\mathbb{Q}}_p$. Assume that F has odd degree or that at some finite place $v \nmid p$ the form f (or more accurately, the associated automorphic representation) is square-integrable. Then ρ_f comes from a GL_2 -type abelian.*

Proof. Due to the hypotheses, one can use the Jacquet-Langlands correspondence to transfer f to a quaternion algebra B over F which splits at precisely one of the infinite places. The result then follows from [Hi, Theorem 4.12], where ρ_f is found in the Jacobian of the Shimura curve associated to B . \square

We now show that one can check if a representation comes from a GL_2 -type abelian variety by passing to a finite extension. The proof of the following lemma comes directly from [Tay3, Corollary 2.4].

Lemma 9.4.4. *Let F be a number field and let $\rho : G_F \rightarrow \operatorname{GL}_2(\overline{\mathbb{Q}}_p)$ be an irreducible representation. Assume there exists a finite extension F'/F such that $\rho|_{G_{F'}}$ is irreducible and comes from a GL_2 -type abelian variety. Then ρ comes from a GL_2 -type abelian variety.*

Proof. Pick a number field K' , an embedding $K' \rightarrow \overline{\mathbb{Q}}_p$ and a $\operatorname{GL}_2(K')$ -type abelian variety A'/F' such that $\rho|_{G_{F'}}$ is equivalent to $T_p A' \otimes_{K'} \overline{\mathbb{Q}}_p$. We have

$$\operatorname{End}_{F, K'}(\operatorname{Res}_F^{F'} A') = \bigoplus_{i \in I} K'_i.$$

Here $\operatorname{End}_{F, K'}$ denotes endomorphisms in the isogeny category of $\operatorname{GL}_2(K')$ -type abelian varieties over F , $\operatorname{Res}_F^{F'} A'$ denotes the restriction of scalars of A' from F' to F , I is some finite index set and each K'_i is a

simple K' -algebra corresponding to some simple direct factor A'_i of $\text{Res}_F^{F'} A'$. Tensoring the above with $\overline{\mathbb{Q}}_p$, we find

$$\text{End}_{F,K'}(\text{Res}_F^{F'} A') \otimes_{K'} \overline{\mathbb{Q}}_p = \bigoplus_{i \in I} (K'_i \otimes_{K'} \overline{\mathbb{Q}}_p)$$

On the other hand, we have

$$\begin{aligned} \text{End}_{F,K'}(\text{Res}_F^{F'} A') \otimes_{K'} \overline{\mathbb{Q}}_p &= \text{End}_{\overline{\mathbb{Q}}_p[G_F]}(T_p(\text{Res}_F^{F'} A') \otimes_{\mathcal{O}_{K'}} \overline{\mathbb{Q}}_p) \\ &= \text{End}_{\overline{\mathbb{Q}}_p[G_F]}(\text{Ind}_{F'}^F(T_p A' \otimes_{\mathcal{O}_{K'}} \overline{\mathbb{Q}}_p)) \\ &= \text{End}_{\overline{\mathbb{Q}}_p[G_F]}(\text{Ind}_{F'}^F(\rho|_{G_{F'}})) \end{aligned}$$

The first step above is Faltings' theorem; the other steps are straightforward. Since $\rho|_{G_{F'}}$ is irreducible the induction $\text{Ind}_{F'}^F(\rho|_{G_{F'}})$ is semi-simple. Furthermore, we have

$$\text{Hom}_{\overline{\mathbb{Q}}_p[G_F]}(\text{Ind}_{F'}^F(\rho|_{G_{F'}}), \rho) = \text{Hom}_{\overline{\mathbb{Q}}_p[G_{F'}]}(\rho|_{G_{F'}}, \rho|_{G_{F'}}) = \overline{\mathbb{Q}}_p.$$

It follows that ρ has multiplicity one in $\text{Ind}_{F'}^F(\rho|_{G_{F'}})$. Thus $\text{End}_{\overline{\mathbb{Q}}_p[G_F]}(\text{Ind}_{F'}^F(\rho|_{G_{F'}}))$ has a canonical $\overline{\mathbb{Q}}_p$ occurring inside of it as a direct factor, namely ρ -isotypic projector. We thus find a canonical $\overline{\mathbb{Q}}_p$ occurring as a summand in $\text{End}_{F,K'}(\text{Res}_F^{F'} A') \otimes_{K'} \overline{\mathbb{Q}}_p$. This factor must appear as a factor of $K'_i \otimes_{K'} \overline{\mathbb{Q}}_p$, for a unique $i \in I$. The simple K' -algebra K'_i must therefore be a field. We take $K = K'_i$, $A = A'_i$ and use the canonical $\overline{\mathbb{Q}}_p$ factor of $K \otimes_{K'} \overline{\mathbb{Q}}_p$ to define our embedding $K \rightarrow \overline{\mathbb{Q}}_p$. It follows directly that ρ is equivalent to $T_p A \otimes_{\mathcal{O}_K} \overline{\mathbb{Q}}_p$ and thus comes from an abelian variety. \square

We now prove Proposition 9.4.1.

Proof of Proposition 9.4.1. Let ρ be given. First consider the case where F has odd degree. Use Theorem 5.1.2 to produce a finite, totally real, Galois extension F''/F which is linearly disjoint from $\ker \bar{\rho}$ and a parallel weight two Hilbert eigenform f'' with coefficients in $\overline{\mathbb{Q}}_p$ such that $\rho|_{G_{F''}} \cong \rho_{f''}$. Let G be the Galois group of F''/F , let H be its 2-Sylow subgroup and let F' be the fixed field of H . The extension F''/F' is Galois with group H and the group H is solvable since it is a 2-group. Thus by solvable descent, we can find a parallel weight two Hilbert modular form f' over F' with coefficients in $\overline{\mathbb{Q}}_p$ such that $\rho|_{G_{F'}} \cong \rho_{f'}$. The field F' has odd degree over F and thus odd degree over \mathbb{Q} . Lemma 9.4.3 now shows that $\rho|_{G_{F'}}$ comes from an abelian variety. Lemma 9.4.4 now gives that ρ comes from an abelian variety.

Now consider the case where F has even degree and ρ is indecomposable at some finite place $v \nmid p$. Apply Proposition 8.2.1 to produce an extension F'/F which is linearly disjoint from $\ker \bar{\rho}$ and in which v splits completely and a parallel weight two Hilbert eigenform f' with coefficients in $\overline{\mathbb{Q}}_p$ such that $\rho|_{G_{F'}} \cong \rho_{f'}$. The form f' is square integrable at v . Lemma 9.4.3 thus shows that $\rho|_{G_{F'}}$ comes from an abelian variety. Lemma 9.4.4 now gives that ρ comes from an abelian variety. \square

We now prove Proposition 9.4.2.

Proof of Proposition 9.4.2. Let $\bar{\rho}$ be given. First assume that $\bar{\rho}$ satisfies (A1). Pick a place $v \nmid p$ of F such that $\bar{\chi}_p|_{G_{F_v}}$ and $\bar{\rho}|_{G_{F_v}}$ are trivial. One easily sees that there is a lift of $\bar{\rho}|_{G_{F_v}}$ which is non-crystalline and of the form

$$\begin{pmatrix} \chi_p & * \\ & 1 \end{pmatrix}$$

Thus $\bar{\rho}$ is compatible with type C at v . Therefore, using Theorem 7.6.1 we can find a finitely ramified, odd, weight two lift ρ of $\bar{\rho}$ which is type C (and thus indecomposable) at v . Proposition 9.4.1 shows that ρ comes from an abelian variety, and so $\bar{\rho}$ does as well.

Now assume that $\bar{\rho}$ does not satisfy (A1). One then finds $\bar{\rho} = \text{Ind}_{F'}^F(\bar{\alpha})$ where F' is the quadratic extension of F contained in $F(\zeta_p)$ and $\bar{\alpha} : G_{F'} \rightarrow \overline{\mathbb{F}}_p^\times$ is some character. Let α be the Teichmüller lift of $\bar{\alpha}$ and put $\rho_0 = \text{Ind}_{F'}^F(\alpha)$. Then ρ_0 is a lift of $\bar{\rho}$. Furthermore, ρ_0 is modular (since it is an induction) and has Hodge-Tate weights zero (since it has finite image). Thus ρ_0 comes from a parallel weight one Hilbert eigenform f_0 over F . Multiply f_0 by a high weight modular form g congruent to 1 modulo p (of the sort provided by [Wi2, Lemma 1.4.2]) and then find a congruence between the resulting form and a parallel weight two eigenform f . If F has odd degree then ρ_f comes from an abelian variety by Lemma 9.4.3, and so $\bar{\rho}$ does as well. If F has even degree, transfer f to the quaternion algebra over F ramified at the infinite places and

use Lemma 4.4.4 to find a congruence between f and a form f' which is special at some finite place prime to p . By Lemma 9.4.3, we find that $\rho_{f'}$ comes from an abelian variety, and so $\bar{\rho}$ does as well. This completes the proof. \square

Remark 9.4.5. Let $\bar{\rho} : G_F \rightarrow \mathrm{GL}_2(\overline{\mathbb{F}}_p)$ be an odd irreducible representation not satisfying (A1). The representation ρ_f constructed in the above proof gives a finitely ramified weight two lift of $\bar{\rho}$. In many cases, one can take the form g to have level divisible only by p ; one can then take f so that ρ_f has the same prime-to- p conductor as $\bar{\rho}$. Thus, in such cases, we can remove the additional hypothesis from the first statement of Theorem 1.1.3.

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